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UNIFIED CONSTRUCTIONS OF THE REGULAR HEPTAGON AND TRISKAIDECAGON

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Abstract

Constructions of regular heptagon and triskaidecagon by trisection of an angle are well known. An elegant construction of the heptagon by S. Adlaj shows a 3-fold symmetry related to a Galois group. Based on the latter construction, in this article one more for the heptagon and two more for the triskaidecagon are presented, all using angle trisection.

Keywords: *geometric construction, trisection, heptagon, triskaidecagon, tridecagon.*

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1. INTRODUCTION

In [1] A. Gleason gave constructions of a regular heptagon and triskaidecagon using only square roots and the trisection of angles. Later S. Adlaj [2] presented a very elegant construction of the heptagon, that shows the action of a cyclic subgroup of order 3 in the related Galois group on the 3 constructed vertices.

In this article I present 3 new unified constructions of the 2 regular polygons by trisection, unified means here *all* based on S. Adlaj's geometric construction. They all use angle trisection.

In section 2 I repeat the construction in [2] and show one more for the heptagon. In section 3 2 constructions for the trikaidecagon are shown.

2. CONSTRUCTIONS OF THE HEPTAGON

2.1. The first construction, type I: S. Adlaj's from [2]

Let $\varepsilon_k = e^{2\pi i/3\,k} = ((-1+\sqrt{3}\,i)/2)^k$, $k = 0,1,2$ a third root of unity. Define the 3 third roots ζ_k, $k = 0,1,2$:

$$\zeta_k = \epsilon_k \sqrt[3]{\zeta}, \qquad \zeta = \frac{1-3\sqrt{3}\,i}{2\sqrt{7}}. \tag{1}$$

Determining the third root of ζ is a equivalent to a trisection because $|\zeta| = 1$. The angle to trisect is $\theta = -\arctan(3\sqrt{3}) \approx -79.1066°$

Take as radii of the 2 grey, concentric circles in figure 1:

$$R_1, R_2 = \sqrt{(7 \pm \sqrt{21})/18}. \tag{2}$$

Form three vertices of the heptagon, the three red, green, blue parallelograms in figure 1 realize the following complex additions:

$$V_0 = R_1\varepsilon_0 + R_2\zeta_1\,, \qquad V_1 = R_1\varepsilon_1 + R_2\zeta_0\,, \qquad V_2 = R_1\varepsilon_2 + R_2\zeta_2\,.^{1} \tag{3}$$

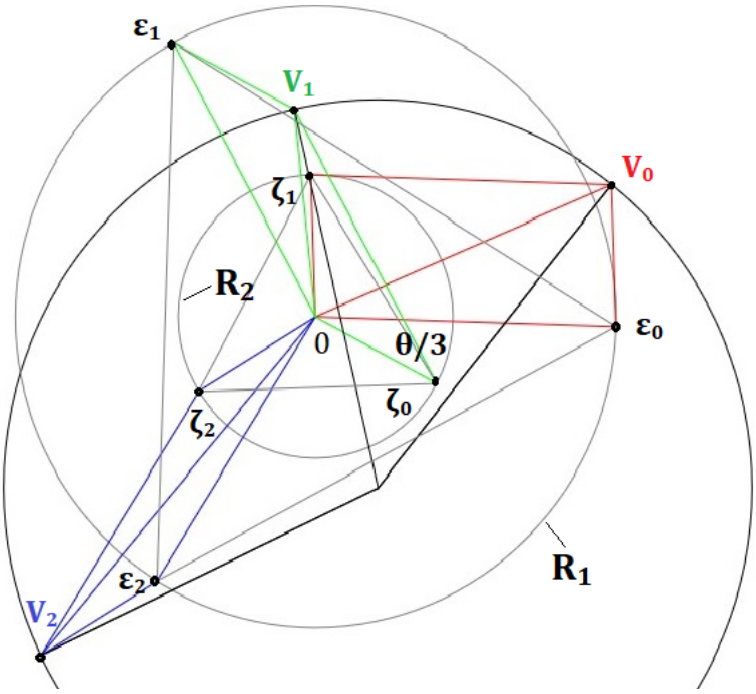

Figure 1. A scan of the heptagon construction in [2] with the denomination of vertices added by the author of this article. The vertices ε_k, ζ_k do not have unit distance to 0, instead they should have been denominated $\varepsilon_k R_1, \zeta_k R_2$

The 3 constructed vertices V_0, V_1, V_2 represent the quadratic residues $PR_2 = \{1,2,4\}$ modulo 7. The triangle built by these 3 vertices has 3 different side lengths and so no symmetry.
The cyclic permutation $C_3 : \zeta_k \to \zeta_{k+1}$ of order 3, subscripts modulo 3, acts on these 3 vertices.

2.2. The second construction, Type II

Now the angle to trisect is $\theta = 0 = 0°$. Only a square root is necesary to get $\theta/3 = 0°, 120°, 240°$.

Set $R_1 = 1$ and R_2 equals one of the six real roots $r_k, k = 0,\ldots,5$ of the palindromic, sextic polynomial $P = r^6 + 6r^5 - 6r^4 - 29r^3 - 6r^2 - 6r + 1$. Because P is palindromic, with every root r, the inverse is a root too. The 6 roots can be constructed by square roots and solving a cubic, because with $Q = (s^3 + 6s^2 - 9s - 41)$ and $s = r + 1/r$, $P = Q(s)\, r^3$.
These 6 roots can also be expressed by the ζ_k gotten by the trisection 1:

$$s_k = -2 + \sqrt{7}\,(\zeta_k + \overline{\zeta_k}), \tag{4}$$

$$r_k, r_{k+3} = \frac{s_k \pm \sqrt{s_k^2 - 4}}{2}, \qquad s_k \ \text{ a root of } \ Q, \qquad k = 0,1,2. \tag{5}$$

[1]There is no typing error here, in V_0, V_1 the ε and ζ subscripts are different

The vertices of the heptagon are given by 3. Hint: a negative R_2 in this geometric construction means: a vector in the parallelogram has to be taken negative.

In contrary to the previous construction, the triangle built by the 3 constructed vertices V_0, V_1, V_2 is now an *isosceles* triangle with a reflection symmetry.

$C_3 : \zeta_k \to \zeta_{k+1}$ acts now on the 6 radii r_k and so on the 6 isosceles triangles. Up to scaling and rotation are only 3 different triangles.

2.3. The third construction, but nothing new, is of type II too

Now the angle to trisect is $\theta = \pi = 180°$. Only a square root is necesary to get $\theta/3 = 60°, 180°, 300°$.

Set $R_1 = 1$ and R_2 equals the *negative* of one of the six real roots r_k of the second construction in 2.2.

Because a negative radius R_2 is equivalent to a positive radius and a rotation by π of the triangle inscribed in this circle: this construction is equivalent to the previous subsection 2.2.

3. CONSTRUCTIONS OF THE TRISKAIDECAGON

3.1. The first construction, type I

Let $\varepsilon_k = e^{2\pi i/3k} = ((-1+\sqrt{3}\,i)/2)^k,\ \ k = 0,1,2$ a third root of unity. Define the 3 third roots $\zeta_k,\ \ k = 0,1,2$:

$$\zeta_k^{\pm} = \varepsilon_k \sqrt[3]{\zeta^{\pm}}, \qquad \zeta^{\pm} = \frac{\sqrt{26 \pm 5\sqrt{13}} - \sqrt{26 \mp 5\sqrt{13}}\,i}{2\sqrt{13}}. \tag{6}$$

Determining the third root of $\zeta^{\pm}$ is a equivalent to a trisection because $|\zeta^{\pm}| = 1$. The 2 corresponding 2 angles to trisect are:

$$\theta^{\pm} = -\arctan\left(\frac{26 \mp 5\sqrt{13}}{3\sqrt{3}\sqrt{13}}\right). \tag{7}$$

Take as radii of the 2 grey, concentric circles and angle to trisect, in figure 1:

$$R_1, R_2 = \sqrt{\frac{\sqrt{13+\sqrt{13}} \pm \sqrt{5+\sqrt{13}}}{2\sqrt{2}}}, \quad R_1 R_2 = 1 \quad \theta^+ \approx -23.0510°. \tag{8}$$

The 3 constructed vertices V_0, V_1, V_2, see 3, represent the now quartic residues $PR_4 = \{1,3,9\}$ modulo 13. Using $-\theta^+/3$ the vertices represent $4\,PR_4 = \{4,10,12\}$. The triangle built by these 3 vertices has 3 different side lengths.

The cyclic permutation $C_3 : \zeta_k \to \zeta_{k+1}$ of order 3, subscripts modulo 3, acts on these 3 vertices.

To get the vertices belonging to the 2 remaing multiplicative cosets of quartic residues $2\,PR_4 = \{2,5,6\}$ and $8\,PR_4 = \{7,8,11\}$, the following 2 radii and angle to trisect have to be used:

$$R_1, R_2 = \sqrt{\frac{\sqrt{13-\sqrt{13}} \pm \sqrt{5-\sqrt{13}}}{2\sqrt{2}}}, \qquad R_1 R_2 = 1, \quad \theta^- \approx -66.9489°. \tag{9}$$

3.2. The second construction, Type II

Now the angle to trisect is $\theta = 0 = 0°$. Only a square root is necesary to get $\theta/3 = 0°, 120°, 240°$.

Set $R_1 = 1$ and R_2 equals one of the 12 real roots $r_k, k = 0, 1, \ldots, 11$ of the palindromic, degree 12 polynomial $P = r^{12} + 12r^{11} - 12r^{10} - 274r^9 - 441r^8 + 441r^7 + 1275r^6 + 441r^5 - 441r^4 - 274r^3 - 12r^2 + 12r + 1$. Because P is palindromic, with every root r, the inverse is a root too. The 12 roots can be constructed by square roots and solving a cubic, because with $Q = (s^6 + 12s^5 - 18s^4 - 334s^3 - 384s^2 + 1323s + 2131)$ and $s = r + 1/r$, $P = Q(s)\, r^6$. The sectic Q factorizes in $\mathbb{Q}(\sqrt{13})$ as product of $R = s^3 + 3(2 + \sqrt{13})\, s^2 + 21(3 + 1\sqrt{13})/2\, s + (15 + 107\sqrt{13})/2)$ and its $\mathbb{Q}(\sqrt{13})$-conjugate $\overline{R}$.

These 12 roots can be expressed by the $\zeta_k^{\pm}$ gotten by the trisection 6:

$$s_k^{\pm} = -(2 \pm \sqrt{13}) + \sqrt{13 \pm \sqrt{13}}\,(\zeta_k^{\pm} + \overline{\zeta_k^{\pm}}), \tag{10}$$

$$r_k, r_{k+3} = \frac{s_k^+ \pm \sqrt{(s_k^+)^2 - 4}}{2}, \qquad s_k^+ \text{ a root of } R, \qquad k = 0, 1, 2, \tag{11}$$

$$r_{k+6}, r_{k+9} = \frac{s_k^- \pm \sqrt{(s_k^-)^2 - 4}}{2}, \qquad s_k^- \text{ a root of } \overline{R}, \qquad k = 0, 1, 2. \tag{12}$$

The conjugation on the rhs of $s_k^{\pm}$ is now a $\mathbb{C}$-conjugation.

The vertices of the trikaidecagon are also given by 3. Hint: a negative R_2 in this geometric construction means: a vector in the parallelogram has to be taken negative.

In contrary to the previous construction, the triangle built by the 3 constructed vertices V_0, V_1, V_2 is now an *isosceles* triangle with a reflection symmetry.

$C_3 : \zeta_k \to \zeta_{k+1}$ acts now on the 12 radii r_k and so on the 12 isosceles triangles. Up to scaling and rotation are only 6 different triangles.

3.3. The third construction, but nothing new, is of type II too

Now the angle to trisect is $\theta = \pi = 180°$. Only a square root is necesary to get $\theta/3 = 60°, 180°, 300°$.

Set $R_1 = 1$ and R_2 equals the *negative* of one of the 12 real roots r_k of the second construction in 3.2.

Because a negative radius R_2 is equivalent to a positive radius and a rotation of the triangle inscribed in this circle: this construction is equivalent to the previous subsection 3.2.

4. CONSTRUCTIONS OF THE PENTAGON

4.1. The first construction, n o type I

A type I construction does not exist. Every triangle built by vertices in a regular pentagon is an isoceles triangle. A triangle with 3 different side lengths for type I does not exist.

4.2. The second construction, Type II

Now the angle to trisect is $\theta = 0 = 0°$. Only a square root is necesary to get $\theta/3 = 0°, 120°, 240°$.

Set $R_1 = 1$ and R_2 equals one of the four real roots $r_k, k = 0, \ldots, 3$ of the palindromic, quartic polynomial $P = r^4 - r^3 - 9r^2 - r + 1$. Because P is palindromic, with every root r, the inverse is a root too. The 4 roots can be constructed by square roots, because with $Q = s^2 - s - 11$ and $s = r + 1/r$, $P = Q(s)\, r^2$. No trisection is needed.

$$s_0, s_1 = (1 \pm 3\sqrt{5})/2 \qquad \text{the 2 roots of } Q \tag{13}$$

$$r_k, r_{k+2} = \frac{s_k \pm \sqrt{s_k^2 - 4}}{2} \qquad k = 0, 1 \tag{14}$$

The vertices of the pentagon are given by 3. Hint: a negative R_2 in this geometric construction means: a vector in the parallelogram has to be taken negative.

The triangle built by the 3 constructed vertices V_0, V_1, V_2 is an *isosceles* triangle with a reflection symmetry.

4.3. The third construction, but nothing new, is of type II too

Now the angle to trisect is $\theta = \pi = 180°$. Only a square root is necesary to get $\theta/3 = 60°, 180°, 300°$.

Set $R_1 = 1$ and R_2 equals the *negative* of one of the four real roots r_k of the second construction in 4.2.

Because a negative radius R_2 is equivalent to a positive radius and a rotation by π of the triangle inscribed in this circle: this construction is equivalent to the previous subsection 4.2.

5. OPEN QUESTIONS

Are the constructions of type I, type II or both also possible for other p-gons with p a prime of the form $6n + 1$? The next primes p to investigate would be $p = 19, 31, \ldots$.

References

1. A. Gleason, "Angle Trisection, the Heptagona, and the Triskaidecagon," *The American Mathematical Monthly*, vol. 95, no. 3, pp. 185–194, 1988; doi:10.2307/2323624
2. S. Adlaj, "The Heptagon constructed!," in *semjonadlaj.com*, 2024. [Online]. Available: https://SemjonAdlaj.com/Excerpts/Heptagon.pdf

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Унифицированные построения правильных семиугольников и тринадцатиугольников

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Аннотация

Построения правильных семиугольника и тринадцатиугольника путем трисекции угла хорошо известны. Элегантная конструкция семиугольника С. Ф. Адлай демонстрирует 3-кратную симметрию, связанную с группой Галуа. Основываясь на этой конструкции, в этой статье мы рассмотрим еще одну для семиугольника и еще две для тринадцатиугольника, все с использованием трисекции угла.

Ключевые слова: *геометрическая конструкция, трисекция, семиугольник, тринадцатиугольник.*

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