algorithmic mathematics and mathematical modeling

Computer tools in education, 2022
№ 3: 64–81
http://cte.eltech.ru
doi:10.32603/2071-2340-2022-3-64-81

COMPUTED MULTIVALUES OF AGM REVEAL PERIODICITIES OF INVERSE FUNCTIONS

Lamarche F.[1], Dr., PhD, ✉ francois.lamarche@teledyne.com
Ruhland H., MSc, Independent Researcher, ✉ helmut.ruhland50@web.de

[1]TeledyneLeCroy, 700 Chestnut Ridge rd., Chestnut Ridge, NY10977 USA

Abstract

The article shows how two choices are possible whenever computing the geometric mean, and the repetition of this process can in general yield 2-to-the power *N* different values when the choices are compounded in the first *N* steps of evaluation of the arithmetic-geometric mean. This happens not only in the simple AGM involved in the computation of the complete elliptic integral of the first kind, but also in analogous methods for the computation of the complete and incomplete elliptic integrals of the first and second kind.

Keywords: *arithmetic, geometric, mean.*

Citation: F. Lamarche and H. Ruhland, "Computed Multivalues of AGM Reveal Periodicities of Inverse Functions," *Computer tools in education*, no. 3, pp. 64–81, 2022; doi: 10.32603/2071-2340-2022-3-64-81

1. INTRODUCTION

Computation of Elliptic integrals with Arithmetic-geometric means has been historically a very powerful tool [1, 2], thanks to its quadratic convergence. Such a convergence was vital before the advent of modern computers with practically unlimited precision available in milliseconds. The duplication formula method represents a modern variation that also exhibits rapid convergence [3]. However, in this article, we choose to focus on the computation of the historic Elliptic Integrals of the First and Second kind, both Complete and Incomplete. The Complete elliptic Integral of the Third kind is expressible as a function of incomplete elliptic integrals of the the First and Second Kind [4]; and we will leave out the computation of the Incomplete Elliptic integral of the Third kind as well as the general case represented by the Carlson symmetric forms — the discussion of their multivaluedness will be the subject of a future article.

Fast processing is one advantage that modern compters provide; large amount of memory to store multiple values is another. In this article, we focus on this aspect, using repeatedly the multivaluedness of the square root function (the key to the geometric mean) to calculate simultaneously multiple values at each iteration of the arithmetic-geometric mean.

Before all, we present the EAGM, which allows to calculate Incomplete and Complete Elliptic Integrals of the first and second kind; then we describe the patterns of multivaluedness of the quantities it produces.

First, we look at the Complete Elliptic Integral of the first kind. Its multivalues span a set that share the property of being quarter periods of the associated Elliptic functions — in a very specific pattern.

Secondly, we will focus on Incomplete Elliptic Integrals of the First kind, whose multivalues possess a regularly repeating pattern with rectangular symmetry on the complex plane. This is found to reflect the double-periodicity of its inverse function, the Jacobian elliptic function sn.

Thirdly, we will look at multivalues of the complete Elliptic integrals of the Second Kind, which make up a pattern related to that of the Complete Elliptic Integrals of the First kind via different scaling of the real and imaginary part.

And fourth, we will look at the multivalues of the Jacobi Zeta function, which represents the non-trivial part of the Incomplete Elliptic function of the Second Kind. The pattern of multivalues found here is very complicated, but a subset reveals an important symmetry of that function, which is related to Weierstrass elliptic function (and to the essential elliptic function [5]).

After the conclusions, in appendices, we provide listings of the matlab functions and study the MAGM algorithm in the context of multivalues.

2. AN EXTENDED AGM, THE EAGM

Here some formulas are given that define a quartet (a_n, g_n, u_n, v_n) which we call EAGM for an obvious reason. The first 2 columns build the classical AGM. This quartet is the base to calculate the incomplete elliptic integrals of the first and second kind.

In each recursion step calculate the following two roots:

$$r_n = \sqrt{a_n g_n}, \qquad s_n = \sqrt{(u_n + v_n)^2 - (a_n - g_n)^2}\Big/2. \tag{1}$$

With these 2 roots calculate in each step the a row of the quartet with:

$$a_{n+1} = \frac{a_n + g_n}{2}, \qquad g_{n+1} = r_n, \qquad u_{n+1} = \frac{u_n + v_n}{2}, \qquad v_{n+1} = s_n. \tag{2}$$

With $\Delta = \sqrt{1 - k^2 \sin^2(\phi)}$ set the initial row of the quartet:

$$(a_0, g_0, u_0, v_0) = \left(1,\ \sqrt{1-k^2},\ 1\big/\sin(\phi),\ \Delta\big/\sin(\phi)\right).$$

With these initial values we get the following identity independent from k and ϕ for all signs of the square roots and all recursion steps n:

$$a_n^2 - g_n^2 = u_n^2 - v_n^2. \tag{3}$$

Writing x_∞ for $\lim\limits_{n\to\infty}(x_n)$, we recognize additional properties:

— the sequences a, g have a common limit, the well known $\mathrm{AGM}(a_0, g_0) = a_\infty = g_\infty$,
— the sequences u, v have a common limit, $u_\infty = v_\infty$,
— for $\phi = \pi/2$ the sequence u, v is a copy of the sequence a, g i.e. $u_n = a_n$ and $v_n = g_n$.

These allow us to calculate the incomplete elliptic integrals of first kind:

$$F(\phi \mid m) = F(\phi \mid k^2) = \sin^{-1}(a_\infty / u_\infty)\big/a_\infty. \tag{4}$$

This quartet also defines a quadratically converging sum for the Jacobi Zeta function:

$$Z(\phi \mid m) = Z(\phi \mid k^2) = \sum_{n=0}^{\infty} 2^n (u_n - v_n)\sqrt{u_n^2 - a_n^2}\Big/u_n. \tag{5}$$

In the equations (1), the sign of the chosen root is not specified. To be more specific, we write:

$$r_n = \sigma_n \sqrt{a_n g_n}, \quad s_n = \delta_n \sqrt{(u_n + v_n)^2 - (a_n - g_n)^2} \Big/ 2. \tag{6}$$

Here is the convention that lets us define the sign unambiguously: For r_n, if σ_n is $+1$, then under this convention, the square root symbol refers to the one root such that the division by $a_n + g_n$ gives a positive real part. So, for $\sigma_n = +1$, we obtain the root and geometric average nearer the arithmetic average, and for $\sigma_n = -1$, we obtain the root and geometric average father from the arithmetic average. When the two root are equally near/far, the positive imaginary part is chosen by convention. Note that taking the "far" root in the calculation of the next AGM hinders convergence, while taking the "near" root in the calculation of the next AGM favors convergence. Note that, in the programs, a variable called "sig g" takes care of flipping the sign in the programs if the real part of the sum is negative.

For s_n, the convention is also geared on convergence. Under this convention, the square root symbol refers to the one of the two roots such that the addition of u_n to this root makes the absolute value larger. Using this convention, δ_n being "+1" will favor convergence toward a finite value, while δ_n being "−1" will hinder convergence to a finite value and instead drive the common limit closer to zero. In the code, we achieve this convention by first taking the square root with a positive real part. Laying out the numbers on the complex plane, if the next u value, u_{n+1}, is within 180 degrees, we don't need to change the sign. Otherwise, in the programs, by setting the variable called "sig" to "−1", we change the sign.

There is yet one more root that can be plus or minus: in (5), we need to pick a convention for the square root sign. By rewriting $u_n^2 - a_n^2$ as $(u_n - a_n)(u_n + a_n)$, we see that we are dealing with geometric mean of $(u_n - a_n)$ and $(u_n + a_n)$, so we can define the square root symbol as yielding the root near the arithmetic average u_n; then γ_n defines if we put a negative sign, and so we represent all the possible choices by writing:

$$Z(\phi \mid m) = Z(\phi \mid k^2) = \sum_{n=0}^{\infty} 2^n (u_n - v_n)\gamma_n \sqrt{u_n^2 - a_n^2} \Big/ u_n. \tag{7}$$

2.1. Analytic continuation and signs of square roots

In this article, we focus on the multivaluedness of complete/incomplete elliptic integrals of first and second kind. Having integrands $I_X(x)$ that are square roots of even rational functions, they are 2-sheeted functions with 4 branch points.

$$X(z) = \int_0^z I_X(x)dx, \quad I_X(x) = \sqrt{\frac{P(x)}{Q(x)}}, \quad P(x), Q(x) \text{ are even quadratic/quartics.} \tag{8}$$

Integration along different closed paths in the plane of the integration variable x can yield different function values at a certain point z in this plane and so result in the multivaluedness. Because the integrand $I_X(x)$ is 2-valued we have 2 different types of closed paths in the integration plane:

- Type 1: the integrand takes the same values at the start and end points of the path.
- Type 2: the integrand takes different values (differing by a factor of −1) at the start and end points of the path, in other words, contrary to type 1, the path is not closed in the 2-fold cover of the integration plane.

The two different types of closed paths lead to two different behaviors of the analytically continued function at a point z:

— for type 1: $X(z) \rightarrow +X(z)+c_1$,
— for type 2: $X(z) \rightarrow -X(z)+c_2$.

The constants c_1, c_2 do not depend on z.

When a multivalued function is given by the limit of a recursion as in the case of $F(k)$ by the AGM or in the case of the Jacobi Zeta function (5) by the recursion (1) and (2), we can expect that analytic continuation along different paths leads to different signs of the roots in the recursion. But not every sign combination has to belong to an analytic continuation along a closed path of the function.

Because the integrand is branched at 4 points, there are two branch cuts, and a type 1 closed path can yield two different values c_1. This causes the multivalues of $X(z)$ to belong to a 2-dimensional lattice, with the generating vectors being the two different values of c_1. In the case of the Jacobi Zeta function, one of the closed paths yields zero, and we get only a 1-dimensional lattice of multivalues.

It is conjectured that for all signs of the roots in r_n, s_n and for all multivalues of $\sin^{-1}()$ in (4) the corresponding function value can be reached by analytic continuation, and that which multivalue of $\sin^{-1}()$ we get corresponds to the type of contour:

— for type 1: $\left(+\sin_p^{-1}(a_\infty/u_\infty)+n\pi\right)/a_\infty$, n even,
— for type 2: $\left(-\sin_p^{-1}(a_\infty/u_\infty)+m\pi\right)/a_\infty$, m odd.

where $\sin_p^{-1}()$ is the principal value of the arc sine function.

3. THE INVERSE OF AGM MAKES A LATTICE ON THE COMPLEX PLANE

We launch the computation of the AGM of unity with the parameter b (also known as $k*$) using the MATLAB program "eagm.m" called by the program "fillk2x32.m" (see appendix A). This MATLAB program, as well as its Excel and python versions, which are attached as documents in the online version, implement a sign choice σ_i, which can be $+1$ or -1, at each of the 5 first iterations, because the numbers 0 thru 31 are represented in binary by just 5 non-zero bits. The sign choices σ_i for $i > 5$ are simply always "$+1$" in order to let convergence occur.

As this program runs, it produces values for $\pi/2/$AGM, shown by colored stars in the complex plane map (fig. 1), which we notice are arranged in a rectangular lattice surrounding and including $K(k)$. Thinking in terms of to a "game of life" [7] analogy, the points are labelled by the number of iterations of non-trivial sign choices needed to reach them, with many new points reached each time the number of free sign choices is incremented by 1 (we stop at '5' here, and one '5' point at $K(k)-64iK(b)$ is not shown as it is out of the plotting area).

Also included in the figure are black points. They too form part of a rectangular lattice, but one that is interleaved vertically between the lattice of colored stars. They are obtained when we start the AGM with unity and $g_0 = -\sqrt{1-k^2}$ instead of $g_0 = \sqrt{1-k^2}$.

3.1. Towards proofs for simple cases: an important relationship

The Landen transformation implies a relationship between $K(k)$ and $K(q)$ where $k = 2\sqrt{q}/(1+q)$:

$$K(k) = 2/(1+b)K(q).$$

with $b = \sqrt{1-k^2}$. This can be combined with the Landen transformation on the complimentary argument $v = \sqrt{1-q^2}$:

$$K(v) = 2/(1+q)K(b)$$

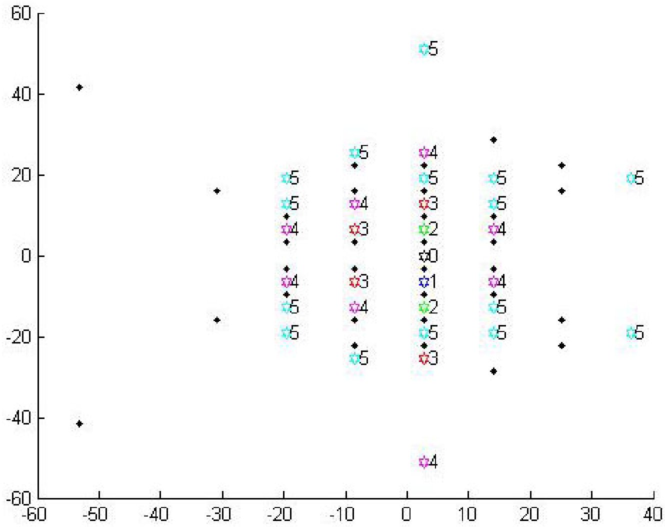

Figure 1. First thirty-two multivalues of $\pi/2/AGM$ on complex plane, for $b = \frac{1}{4}$, and for $b = -\frac{1}{4}$

to yield this relationship between the values of elliptic integrals:

$$K(k)K(v) = 2K(b)K(q).$$

This result is because $(1+b)(1+q)$ is equal to two, as can be seen by expanding $q = \sqrt{1-v^2} = \sqrt{1-(2\sqrt{b}/(1+b))^2} = (1+b)^{-1}\sqrt{(1+b)^2-4b} = (1+b)^{-1}(1-b)$, which means that $1+q = (1+b)^{-1}(1+b+1-b) = 2/(1+b)$. Note that such a relationship can be extended over a second transformation; writing $q = 2\sqrt{c}/(1+c)$ and $w = 2\sqrt{v}/(1+v)$, we have:

$$K(k) = \frac{2}{1+b}\frac{2}{1+v}K(c)$$

and

$$K(w) = \frac{2}{1+c}\frac{2}{1+q}K(b)$$

and so

$$K(k)K(w) = 4K(b)K(c).$$

With this, we are now ready to perform an exact calculation of the effect of different sign choices on $K(k)$. But first, we must note that there exist two ways to assess the multivaluedness of $K(k)$ as calculated via $\mathrm{AGM}(1,b)$. The first way is to consider that both b and $-b$ are solutions to $k^2+b^2=1$. This multiplies our overall sign choices by two, and, not surprisingly, it creates a denser lattice, with lattice parameters $4K(k)$ and $2iK(b)$.

3.2. A first way to calculate multivalued $K(k)$

For this, we just look at what $K(k) = \pi/2/\mathrm{AGM}(1,b)$ becomes if we choose $b_- = -\sqrt{1-k^2}$ for b. The denominator becomes

$$\mathrm{AGM}(1,b_-) = \mathrm{AGM}\left(\frac{1+b_-}{2}, \sqrt{b_-}\right) = \frac{1-b}{2}\mathrm{AGM}\left(1, 2i\frac{\sqrt{b}}{1-b}\right)$$

we can then use the definition of K to write it as:

$$\mathrm{AGM}(1,b_-) = \frac{1-b}{2}\frac{\pi}{2}\Big/K\left(\sqrt{1-(2i\frac{\sqrt{b}}{1-b})^2}\right) = \frac{1-b}{2}\frac{\pi}{2}\Big/K\left(\sqrt{1+4b/(1-b)^2)}\right) =$$

$$= \frac{1-b}{2}\frac{\pi}{2}\Big/K\left(\frac{1+b}{1-b}\right).$$

Known formulae [6] allow us to compute K for k values greater than 1; these yield:

$$\mathrm{AGM}(1,b_-) = \frac{1+b}{2}\frac{\pi}{2}\Big/\left(K\left(\frac{1-b}{1+b}\right) \mp iK\left(\sqrt{1-\left(\frac{1-b}{1+b}\right)^2}\right)\right).$$

Using the variables introduced for the transformed value:

$$\mathrm{AGM}(1,b_-) = \frac{1+b}{2}\frac{\pi}{2}\Big/\big(K(q) \mp iK(v)\big).$$

Finally, the ratio $\pi/2\big/\mathrm{AGM}(1,b_-)$ can be finally written as:

$$\frac{2}{1+b}K(q) \mp 2i\frac{1+q}{2}K(v) = K(k) \mp 2iK(b).$$

The distance between $K(k)$ and this value is $2iK(b)$, showing that the smallest complex numbers of the second lattice ($-b$ instead of b) are offset by $2iK(b)$.

3.3. A second, stricter way to calculate multivalued $K(k)$

We look at what happens if we decide the sign of b is not changeable. The denominator can still be called $\mathrm{AGM}(1,b)$, but in the calculation of the AGM, we choose the negative sign instead of the positive sign when calculating some of the geometric means. We denote this by $\mathrm{AGM}_{-,+,\ldots}$, where we indicate the chosen sign at each iteration, and the ellipsis (...) denotes a indefinite repetition of the last sign choice. As the computer algorithm runs, we explore many such sign choices, but in here, we will just explore what happens with only the first sign flipped; in this case, the denominator is

$$\mathrm{AGM}_{-,+,\ldots}(1,b) = \mathrm{AGM}\left(\frac{1+b}{2}, -\sqrt{b}\right) = \frac{1+b-2\sqrt{b}}{4}\mathrm{AGM}\left(1, \frac{\sqrt{(b+1)\sqrt{b}}}{1+b-2\sqrt{b}}i\right).$$

This is again a call for an AGM with an imaginary number, corresponding to an evaluation of the complete elliptic integral of the first kind with an argument greater than one, and so we again use the formula [6]:

$$\mathrm{AGM}_{-,+,\ldots}(1,b) = \frac{1+b}{2}\frac{1-v}{2}\frac{\pi}{2}\Big/K\Big(\frac{1+v}{1-v}\Big)$$

is

$$\frac{1+b}{2}\frac{1+v}{2}\frac{\pi}{2}\Big/\left(K\Big(\frac{1-v}{1+v}\Big) \mp iK\left(\sqrt{1-\Big(\frac{1-v}{1+v}\Big)^2}\right)\right)$$

and that is

$$\mathrm{AGM}_{-,+,\ldots}(1,b) = \frac{1+b}{2}\frac{1+v}{2}\frac{\pi}{2}\Big/\big(K(c) \mp iK(w)\big) = \frac{\pi}{2}\Big/\big(K(k) \mp 4iK(b)\big).$$

What happens is the factor needed for returning $K(w)$ to $K(b)$ is just the same as the factor needed to return $K(w)$ into $K(b)$, except for the factor of four as shown above. The difference between the scaled inverses of $\mathrm{AGM}(1,b)$ and $\mathrm{AGM}_{-,+,\ldots}(1,b)$ is $4iK(b)$, showing an example of the second lattice parameter being $4iK(b)$ (which is also the second period of all elliptic functions of parameter k).

3.4. Important observations and conjectures about the values obtained by a finite number of sign choices

In general, the choice of a square root of a product of two values can be divided between a square root which is nearer to the two values and one which is farther. The exception is when the two values are exactly 180 degrees apart, i.e. opposite sign. In that case, one of the two roots (specifically, the root with a positive imaginary part) is arbitrarily chosen as the nearest; for this reason, the pattern of stars does not have perfect up-down symmetry.

The population of values obtained by considering more and more applications of different sign choices appear to constitute sort of a “game of life” [7], albeit with very different rules. In contrast with the classical game, at each generation, the previously filled lattice points remain present. At each step, there are new multivalues, and they could in principle double the number of multivalues at each step. The population of multivalues will exactly double at each iteration of our zero-player game if there are no repeat values. So far, it seems no point of the lattice exists repeatedly among the 2^n sign combinations, but we do not yet have a proof of that conjecture.

The pattern of multivalues with up to N (in our example 5) arbitrary sign choices seems to produce values up to 2^{N-1} (in our example, $2^4 = 16$) unit lattice steps away. The extent of the cloud of multivalues thus grows exponentially; however, it seems that some values on this lattice are never reached, such as non-trivial lattice points with zero imaginary part. It is not yet known if all the non-zero imaginary part lattice points can be reached given a sufficient number of iterations of the sign choices.

It is speculated that a formal proof of the multivalue being exactly a lattice point could be developed automatically by a symbolic calculation generator, but we have not yet attempted to develop such an automated proof builder.

The values of $\pi/2\big/\mathrm{AGM}$ reached by positive and negative values of b are very different, even though they both correspond to the same $k = \sqrt{1-b^2}$ argument of the elliptic integral K. This suggests suggests that, in a sense, the function $\kappa(b) = \pi/2\big/\mathrm{AGM}(1,b)$ is more fundamental than $K(k)$.

The observations and conjectures about the numbered symbols in our graphic also apply to the black dots (where we start with $-b$); except those multivalues appears to have exact up-down symmetry.

4. MULTIVALUE LATTICE FOR THE INCOMPLETE INTEGRAL OF THE FIRST KIND

We again launch the computation of the EAGM of unity with the parameter b (also known as $k*$) using the MATLAB program “eagm.m” called by the program “fillf8x16.m” (see appendix A). The EAGM has a second parameter, ϕ, that controls the amount of incompleteness. To illustrate the working of this multivalue-computing program, we choose a value of b of 0.25 and a value of $\sin\phi$ of 0.8. We allow the sign of the geometric average σ_n to be +(the “near” geometric average) or — (the "far" geometric average) during the first 3 iterations; it is always chosen "+" after that, yielding convergence to a non-zero value. We further allow the sign δ_n to be + or − during the first 4 iterations, and for the remaining iterations it is always "+".

As this program runs, it produces values for $\arcsin z/\mathrm{AGM}$, shown by blue stars in the complex plane map (fig. 2), which we can notice are arranged in a rectangular lattice with the same generating vectors as $K(k)$.

We observe that this lattice pattern comprises two multi-values per cell.

We observe that sometimes, the same multivalues can be reached by different choices of σ_n and δ_n.

We make a conjecture that every value on the lattice will be reached by sign combinations having a finite number of "−" choices.

Even if not all values on this lattice of multivalues could be reached within a finite number of "−" sign choices, the $\arcsin z$ function has branches expressible as $\pi - \arcsin z$, as $2\pi + \arcsin z$, etc... and those will definitely fill the lattice. The reason is the multivaluedness of $K = \pi/2\big/\mathrm{AGM}$ having the same lattice generating vectors as those of F.

The lattice of values reflects the double periodicity of the inverse function of F: namely, the double-periodicity of the Jacobian elliptic function $sn(z)$. But while the lattice generating values of our F multivalues are $4K(k)$ and $4iK(b)$, the periods of sn are $4K(k)$ and $2iK(b)$. However, when considering all functions together, the Jacobi elliptic functions have periods $4K(k)$ and $4iK(b)$.

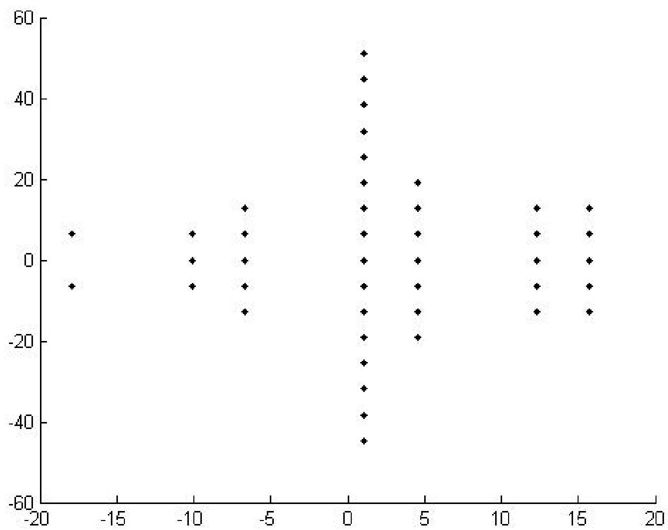

Figure 2. First 128 (8 by 16) multivalues of multivalued $F(\sin\phi, k)$ on complex plane, for $b{=}\frac{1}{4}$ and $\sin\phi = 0.8$

5. MULTIVALUE LATTICE FOR THE COMPLETE INTEGRAL OF THE SECOND KIND

Following [2], the computation of the AGM can be accompanied by a summation of the square differences between the arithmetic and geometric means. The square differences factor in the sum is not constant, but raises by a factor of two at each iteration.

These sums are then divided by the AGM. This process is repeated in much the same way as the computation of the inverses of the AGM over choices of the sign of the geometric mean.

The result, for four choices of the sign of the geometric mean (see “fille32” in appendix A), is a set of mutivalues whose complex plane map is shown in figure 3.

We observe that it has the exact same shape as the set of section 3.

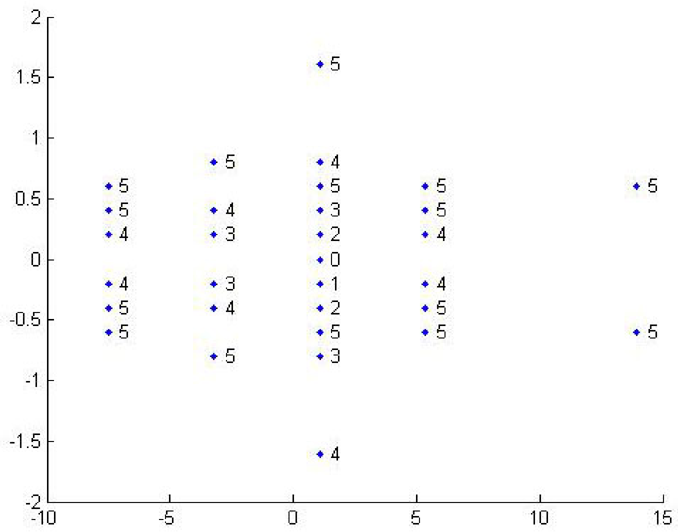

Figure 3. First thirty-two multivalues of multivalued *E(k)* on complex plane, for $b = \frac{1}{4}$

While the shape, in terms of reached integer values of the lattice generating vectors, are the same, the generating vector are of different sizes.

The horizontal/real factor is obvious: $E(k)/K(k)$, because the real-axis lattice generating vector is $4E(k)$ instead of $4K(k)$.

The vertical/imaginary factor is $\big(K(b)-E(b)\big)/K(b)$, because the imaginary-axis lattice generating vector is $4iK(b)-4iE(b)$ instead of $4iK(b)$. The proof of this ratio can be obtained by using the development of the values of the AGM for alternate sign choices, using the same methods used to develop K in the subsections of section 3, and similar known formulae [6]. The consequence of the unequal scaling is that the ratio between $E(k)$ and $K(k)$, called $N(b^2) = N(1-k^2)$, itself will have multivalues, and those multivalues will be located on a circle on the complex plane. The plot of $N(b^2)$ for 32 multivalues runs in “filln32” program and yields figure 4. It is remarkable that $N(b^2)$ forms a dense set even though it is a set built from the ratio of two complex numbers each having a lattice extending to infinity in all directions.

In fact, it is interesting to note that for any set of points on the complex plane, taking the ratio point-wise between such a scaled version of this set and the original set will yield a circle, if the scaling is different in the real and imaginary directions.

6. MULTIVALUE LATTICES RELATING TO THE INCOMPLETE INTEGRAL OF THE SECOND KIND

The incomplete elliptic integral of the second kind is usually calculated splitting it into a trivial part, proportional to the incomplete elliptic of the first kind, plus a function called Zeta:

$$E\big(\sin(\phi),k\big) = F\big(\sin(\phi),k\big)E(k)/K(k) + Z\big(\sin(\phi),k\big).$$

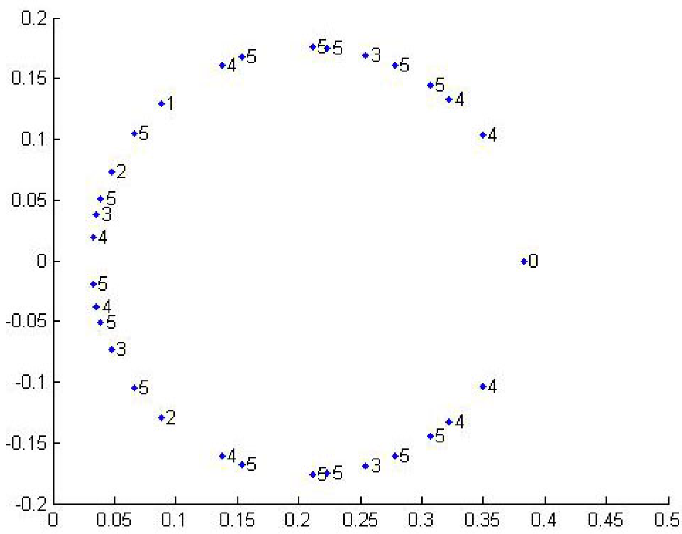

Figure 4. First thirty-two multivalues of the multivalued MAGM function $N(b^2)$ on complex plane, for $b=\frac{1}{4}$

Just like the computation of $E(k)$, the computation of $Z\left(\sin(\phi), k\right)$ involves a sum of terms, each one multiplied by a successively larger power of two.

When we calculate the effect of taking any choice of sign σ_n, δ_n and γ_n, a very chaotic pattern emerges for Z on the complex plane. See figure 5.

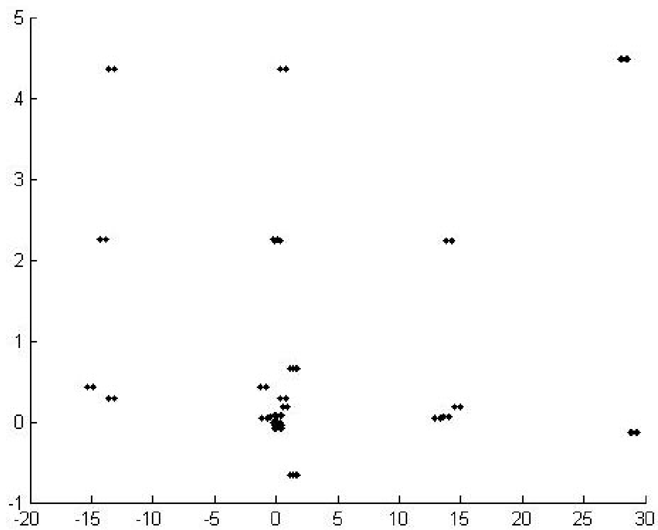

Figure 5. First 64 (4 by 4 by 4) multivalues of multivalued $Z(\sin\phi, k)$ on complex plane, for $b = \frac{1}{4}$ and $\sin\phi = 0.8$

However, if we restrict the choice of sign to δ_n,and make the sign γ correlated to δ, with $\gamma_n = \delta_{n-1}$, we find that the pattern of multivalues has regularity — it forms a one-dimensional lattice, see the other figure 6.

The existence of this pattern reflects a property of the Z function, namely that after crossing two branch cuts, it comes back to the same value as before, except for the addition of an imaginary constant. In the following paragraphs, we derive this imaginary constant.

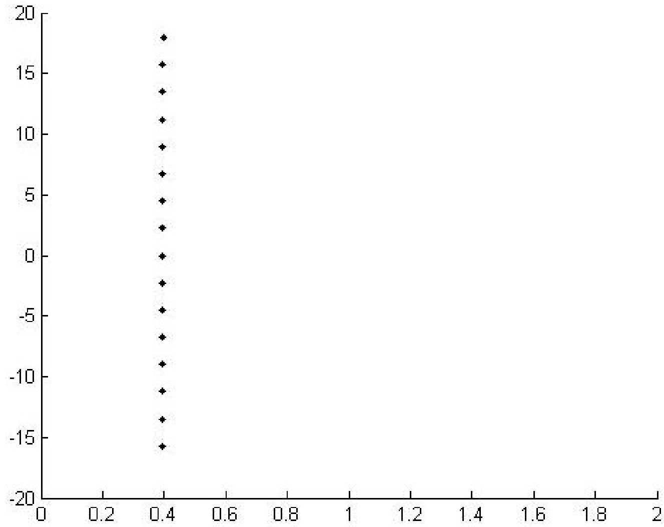

Figure 6. First 16 restricted multivalues of multivalued $Z(sin(\phi), k)$ on complex plane, for $b = \frac{1}{4}$ and $\sin\phi = 0.8$

6.1. Calculation of the multi-valuedness of Jacobi zeta function.

For ease, we will denote by b the function $\sqrt{1-k^2}$ of the elliptic integral parameter k, and we may also sometimes use m to denote k^2.

The Jacobi zeta function is multivalued, and the values are separated by a "lattice constant" which can be evaluated. An inverse function of the Jacobi Zeta function would have this "lattice constant" as periodicity. In order to calculate the separation between multiple values of this function, we use its definition as

$$E(\phi, k) - E(k)F(\phi, k)/K(k)$$

and calculate the change of the function as it crosses a first branch cut, going around the branch point at $\pi/2 + i\sinh^{-1}(b/k)$ or $\pi/2 + i\cosh^{-1}(1/k)$, followed a second branch cut, going around the branch point at the complex conjugate position $\pi/2 - i\sinh^{-1}(b/k)$.

We use the [8] reference to measure the change of the function F at the branch cut; this change consists of the addition of this constant C_F, along with a switch to the $-F$ function, so as to maintain continuity. When going around the other branch cut, the change consists of the addition of the same constant, along with a switch back to the F function.

$$C_F = 4K(k) - \frac{2}{k}K\left(\frac{1}{k}\right).$$

Similarly, we use the [9] reference evaluate the change of the function E after going around the two branch cuts, it is two times this other constant:

$$C_E = 4E(k) - 2k\left(E\left(\frac{1}{k}\right) + \left(\frac{1}{k^2} - 1\right)K\left(\frac{1}{k}\right)\right).$$

By definition of Z, the effect on Jacobi Z of going around the two branch points is the addition of a constant $Q_Z = 2C_E - 2C_F E(k)/K(k)$. One immediately notices that the $4E(k)$ term cancels the scaled $4K(k)$ term. Then one is left with...

$$Q_Z = 2C_E - 2C_F \frac{E(k)}{K(k)} = -4kE\left(\frac{1}{k}\right) + 4kK\left(\frac{1}{k}\right) - \frac{4}{k}K\left(\frac{1}{k}\right) + \frac{4}{k}\frac{E(k)K(1/k)}{K(k)} =$$
$$= -4kE\left(\frac{1}{k}\right) + \left(4k - \frac{4}{k} + \frac{4}{k}\frac{E(k)}{K(k)}\right)K\left(\frac{1}{k}\right).$$

We further use [6] to develop the values of the elliptic integral with parameter greater than one into real and imaginary parts, yielding this expression for Q_Z:

$$-4E(k) \mp 4iE(b) + 4(1-k^2)K(k) \pm 4ik^2K(b) + \left(4k^2 - 4 + 4\frac{E(k)}{K(k)}\right)K(k) \mp i\left(4k^2 - 4 + 4\frac{E(k)}{K(k)}\right)K(b).$$

The real parts of Q_Z cancel out, and it leaves out:

$$Q_Z = \mp 4iE(b) \pm 4ik^2K(b) \mp 4ik^2K(b) \pm 4iK(b) \mp 4iE(k)K(b)\big/K(k)$$

or, removing canceling terms,

$$Q_Z = \mp 4iE(b) \pm 4iK(b) \mp 4iE(k)K(b)\big/K(k)$$

or, grouping the denominator,

$$Q_Z = \mp 4i\big(E(b)K(k) - K(b)K(k) + E(k)K(b)\big)\big/K(k)$$

and so finally, using the Legendre's relation

$$Q_Z = \mp 4i\frac{\pi}{2}\big/K(k) = \mp\frac{2\pi i}{K(k)}.$$

6.2. Significance of the multivaluedness of Zeta

Since the multivaluedness (of the 2D lattice kind) of F is linked to the (double) periodicity of its inverse function, snu, one can wonder if the multivaluedness of Z in a regular pattern is indicative of a periodicity of its inverse function. It is relatively easy to see that it is not the case, at least in the most basic meaning of periodicity. If we take the example $Z(0.5, \sqrt{0.9375}) \approx 0.2920$, meaning the inverse function ξ has a value $\xi(0.2920, \sqrt{0.9375}) \approx 0.5$, it is easy to solve numerically and see that $\xi(0.2920+2.2430i, \sqrt{0.9375}) \approx 0.3150+2.7672i \neq 0.5$ because $Z(0.3150+2.7672i, \sqrt{0.9375}) \approx 0.2920 + 2.2430i$. So $2.2430i$, the unit vector of the 1D lattice of multivalues, is not a period of ξ the inverse function of Z.

However, just as we were able to analytically continue Z around a branch cut, its inverse function ξ could be analytically continued following Z around a branch cut. After following it around two branch cuts (i.e. an even number of times, to restore the original sign), we would arrive to an offset version of ξ, which reaches the target value of 0.5 at the precise offset corresponding to the "period". We have a kind of periodicity, but it is only when considering ξ and all of its "branches".

7. CONCLUSIONS

We have shown that using the sign choice ambiguity in AGM calculation yields multivalues that hint at the multivaluedness of the functions we call elliptic integrals. This multivaluedness spans a lattice on the complex plane which itself points at the double periodicity of the inverse of these functions, the Jacobian elliptic functions. In the case of the incomplete elliptic integral of the second kind, we can only give meaning to some of the multivalues, so many challenges remain.

APPENDICES

Appendix A: program to calculate multivalues

```
function [zd,f,kn,ko]=eagm(k,sinphi,signb,sign1s,sign2s,sign3s)
    n0=1;
    g=signb*sqrt(1-k*k);
    a=1;
    u=1/sinphi;
    v=signb*sqrt(1-k*k*sinphi*sinphi)/sinphi;
    zd=0;
    p2=1;
    for j=0:19
        n0=n0-(a*a-g*g)*p2/2;
        if (real(sqrt(z*z-a*a)/z)>=0)
            sig_z=1;
        else
            sig_z=-1;
        end
        if (bitand(2^j,sign3s))
            zd=zd-sig_z*(z-v)/z*sqrt(z*z-a*a)*p2;
        else
            zd=zd+sig_z*(z-v)/z*sqrt(z*z-a*a)*p2;
        end
        t=sqrt(a*g);
        if real(t/(a+g))>=0
            sig_g=1;
        else
            sig_g=-1;
        end
        if bitand(2^j,sign1s) % this integerbit code selects far vs near
            t=-sig_g*sqrt(a*g); % the far geometric mean
        else
            t=sig_g*sqrt(a*g);  % the near geometric mean
        end

        vo=v;
        v=sqrt((u+vo)*(u+vo)-(a-g)*(a-g))/2;
        if real(v/(u+vo))>=0   % sig*v is the root that allows convergence
            sig=1;
        else
            sig=-1;
        end
        if (bitand(2^j,sign2s))
            v=-v*sig;
        else
            v=+v*sig;
        end
        u=u/2+vo/2;
        a=a/2+g/2;
        g=t;
        p2=p2*2;
    end
```

```
    f=asin(a/u)/a;
    ko=pi/2/a;
    kn=ko*n0;
end
% fillk2x32
function fillk2x32(k)
if ~exist('k')
    k=sqrt(0.9375);
end
colormap='kbgrmc';
hold on
for jj=31:-1:0
for ll=0:-1:0
for mm=0:-1:0
    if (jj~=16)
        [zd,ff,ee,kc]=eagm(sqrt(0.9375),0.5,1,jj,ll,mm);
        generation=1+floor(log(jj+0.99)/log(2));
        plot(real(kc),imag(kc),strcat(colormap(1+generation),'h'));
        text(real(kc)+0.8,imag(kc),num2str(generation));
    else
        display 'one point is skipped because it suffers from numeric
        precision limitations and it would change too much the scale of the plot'
    end
    [zd,ff,ee,kc]=eagm(sqrt(0.9375),0.5,-1,jj,ll,mm);
    plot(real(kc),imag(kc),'k.');
end
end
end
hold off
end
% fillf8x16
function fillf8x16(sinphi,k)
if ~exist('k')
    k=sqrt(0.9375);
end
if ~exist('sinphi')
    sinphi=0.8;
end
hold on
for jj=7:-1:0
for ll=15:-1:0
for mm=0:-1:0
    [zd,ff,ee,kc]=eagm(k,sinphi,1,jj,ll,mm);
    plot(real(ff),imag(ff),'k.')
end
end
end
hold off
end
% fille32
function fille32(k)
if ~exist('k')
    k=sqrt(0.9375); % example value by defaut
```

```
end
hold on
for jj=31:-1:0
for ll=0:-1:0
for mm=0:-1:0
    if (jj~=16)
        [zd,ff,ee,kc]=eagm(k,0.5,1,jj,ll,mm);
        generation=1+floor(log(jj+0.99)/log(2));
        plot(real(ee),imag(ee),'b.');
        text(real(ee)+0.5,imag(ee),num2str(generation));
    else
        display 'one point is skipped because it suffers from numeric
        precision limitations and it would change too much the scale of the plot'
    end
end
end
end
hold off
end
% filln32
function filln32(k)
if ~exist('k')
    k=sqrt(0.9375); % example value by defaut
end
hold on
for jj=31:-1:0
for ll=0:-1:0
for mm=0:-1:0
    if (jj~=16)
        [zd,ff,ee,kc]=eagm(k,0.5,1,jj,ll,mm);
        generation=1+floor(log(jj+0.99)/log(2));
        plot(real(ee/kc),imag(ee/kc),'b.');
        text(real(ee/kc)+0.005,imag(ee/kc),num2str(generation));
    else
        display 'one point is skipped because it suffers from numeric
        precision limitations and it would change too much the scale of the plot'
    end
end
end
end
axis([0 0.5 -0.2 0.2])
hold off
end
% fillz4x4x4
function fillz4x4x4(sinphi,k)
if ~exist('k')
    k=sqrt(0.9375);
end
if ~exist('sinphi')
    sinphi=0.8;
end
hold on
for jj=3:-1:0
```

```
for ll=3:-1:0
for mm=3:-1:0
    [zd,ff,ee,kc]=eagm(sqrt(0.9375),0.5,1,jj,ll,mm);
    plot(real(zd),imag(zd),'k.')
end
end
end
hold off
end
%fillz1x16
function fillz1x16(k,sinphi)
if ~exist('k')
    k=sqrt(0.9375);
end
if ~exist('sinphi')
    sinphi=0.8;
end
hold on
for jj=0:-1:0
for ll=15:-1:0
for mm=0:-1:0
    [zd,ff,ee,kc]=eagm2(k,sinphi,1,jj,kk,ll,ll*2);
    plot(real(zd),imag(zd),'k.')
end
end
end
hold off
axis([0 2 -20 20])
end
```

Appendix B: Relationship with the MAGM

In this appendix, we will explore another algorithm involving repeatedly taking a square root: the MAGM algorithm discovered by S. Adlaj [10]. First, considering only positive choices of the square root, we prove that that algorithm is equivalent to the algorithm discovered by Gauss (and exposed very clearly in many review articles, such as Jameson [2], and used in our EAGM code). Then we will see this method only works for this choice of square root signs.

Demonstration of the equivalence of the MAGM with the Gauss method

Starting with

$$a_{n+1} = \frac{a_n + g_n}{2}$$

and

$$g_{n+1} = \sqrt{a_n g_n}$$

we note that

$$\frac{E(\sqrt{1-b^2})}{K(\sqrt{1-b^2})} = 1 - S$$

where

$$S = \sum_{n=0}^{\infty} 2^{n-1}\left(a_n^2 - g_n^2\right).$$

It is easy to see that the first few rows of MAGM calculation correspond exactly to the partial evaluations of S by directly evaluating them explicitly.

Table 1. First rows of AGM and MAGM evaluation

row index	arithmetic mean	geometric mean	current series	MAGM x	MAGM y	MAGM z
i	a_i	g_i	$1-S$	x_i	y_i	z_i
0	1	b	1	1	b^2	0
1	$\frac{1+b}{2}$	$\sqrt{b}$	$\frac{1}{2}+\frac{b^2}{2}$	$\frac{1}{2}+\frac{b^2}{2}$	b	$-b$
2	$\frac{1+2\sqrt{b}+b}{4}$	$\sqrt{\frac{1+b}{2}\sqrt{b}}$	$\frac{(1+b)^2}{4}$	$\frac{(1+b)^2}{4}$	$\sqrt{b}(1+b)-b$	$-\sqrt{b}(1+b)-b$
…	…	…	…	…	…	…
$n+1$	$\frac{a_n+g_n}{2}$	$\sqrt{a_n g_n}$	…	$\frac{x_n+y_n}{2}$	$z_n+\sqrt{(x_n-z_n)(y_n-z_n)}$	$2z_n-y_{n+1}$

In order to establish that this correspondence continues indefinitely, it is possible to make an *ad absurdum* demonstration: we start by assuming that the correspondence stops being true at some row B, and use the properties of the AGM and MAGM to deduce that the correspondence will stop being true at row $B-1$.

First, note that for the correspondence between the series and the MAGM to continue up to row B, all we need is for $x_{n+1}-x_n$ to be equal to $2^{n-1}(a_n^2-g_n^2)$ for $n \le B$.

Then note that the scaling invariance of the AGM means we can divide every (a_n, g_n) pair by $\frac{1+b}{2}$ and still have an AGM sequence of pairs of numbers.

The translational invarience of the MAGM means we can add b to every triplet, the resulting sequence of triplets still being MAGM.

And note that the scaling invariance of the MAGM means we can multiply every (x_n, y_n, z_n) triplet by $\frac{(1+b)^2}{2}$ and still have a MAGM sequence of triplets.

If we change the origin of indexing of the AGM by one, then the consecutive terms of S each change by a factor of two, because of the exponent of 2 in each term $2^{n-1}(a_n^2-g_n^2)$ changes by one.

Combining the scalings, translation, and index origin change by one, we note that the second row effectively becomes a first row, only with the symbol b replaced by a symbol v, where $v=\frac{2\sqrt{b}}{1+b}$.

We get the impossible conclusion that the correspondence stops being true at $B-1$ as far as v is considered, whereas it stopped being true at B as far as b is considered. Therefore, this correspondence must continue infinitely.

What about $g_{n+1} = -\sqrt{a_n g_n}$ at specific steps in the calculation

We ran the MAGM, letting the sign of the square root of $(x_n-z_n)(y_n-z_n)$ be negative for various n (combined with many choice combinations of σ_n), and we did not obtain any good multivalues of $E(k)$.

Thinking about the above demonstration, where we transform a second row into a first row, such a transformation is only possible if all the sign choices for the square root are the same, positive or negative.

All the signs being negative never yields convergence, so it is no wonder that the MAGM algorithm only delivers $E(k)$ for the “all positive sign choices” case.

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Received 26.09.2022, the final version — 31.10.2022.

Francois Lamarche, Dr., PhD, TeledyneLeCroy, USA, ✉ francois.lamarche@teledyne.com

Helmut Ruhland, MSc, Independent Researcher, Santa Fé, La Habana, Cuba, ✉ helmut.ruhland50@web.de

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Компьютерные инструменты в образовании, 2022

№ 3: 64–81

УДК: 517.16:517.18

http://cte.eltech.ru

doi:10.32603/2071-2340-2022-3-64-81

Вычисленные множественные значения АГС выявляют периодичность обратных функций

Ламарш Ф.[1], Doctor, PhD, ✉ francois.lamarche@teledyne.com
Руланд Х., MSc, независимый исследователь, ✉ helmut.ruhland50@web.de

[1] Корпорация Теледайн ЛеКрой, 700 Честнат-Ридж роуд, Честнат-Ридж, NY10977 США

Аннотация

В статье показано, как возможны два выбора при вычислении значения геометрического среднего, и повторение такой процедуры на первых *N* шагах вычисления арифметико-геометрического среднего может в целом давать 2 в степени *N* различных значений, когда соответствующие выборы комбинируются. Это происходит не только в простом АГС, при вычислении полного эллиптического интеграла первого рода, но и в аналогичных методах при вычислении полных и неполных эллиптических интегралов первого и второго рода.

Ключевые слова: *арифметическое среднее, геометрическое среднее.*

Цитирование: Ламарш Ф., Руланд Х. Вычисленные многозначности АГС выявляют периодичность обратных функций // Компьютерные инструменты в образовании. 2022. № 3. С. 64–81. doi: 10.32603/2071-2340-2022-3-64-81

Поступила в редакцию 26.09.2022, окончательный вариант — 31.10.2022.

Франсуа Ламарш, Doctor, PhD, метролог, корпорация Теледайн ЛеКрой, Честнат-Ридж, США, ✉ francois.lamarche@teledyne.com

Хельмут Руланд, MSc, независимый исследователь, Санта-Фе, Гавана, Куба, ✉ helmut.ruhland50@web.de