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RESOLUTION OF SUDOKUS USING GROEBNER BASIS

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Abstract

We study the resolution of sudokus and generalized sudokus using Groebner basis. Let $x_1,\ldots,x_{81}$ the 81 squares which form the sudoku, arranged from left to right and from top to bottom. Its solution will be $(a_1,\ldots,a_{81})$, where a_i is the number in the square associated to the variable x_i.

Let S be a sudoku with preassigned data $\{c_i\}_{i\in L}$, for $L\subset\{1,\ldots,81\}$. All the necessary information to solve the sudoku is contained in the algebraic set $\mathbb{V}(I+<\{x_i-c_i\}_{i\in L}>)$. We shall use Groebner basis to find a solution and give a SAGE code for that purpose.

Keywords: *Sudoku, generalized sudoku, Gröbner base, algebraic manifold.*

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1. INTRODUCTION

A **sudoku** puzzle is a 9×9 grid divided into nine 3×3 boxes where there are numbers between 1 and 9 in some of the squares in the grid. To solve the puzzle we have to find the remaining numbers in such a way that every row and every column only contain the digits 1 to 9 and also every 3×3 box only contains those digits, without repetitions.The puzzle derives its name from the japanese words Su, which means number and Doku, which means solitary. This puzzle was popular in Japan since 1986 and it was in 2005 that it became internationally known. This mathematical game had it origin in New York at the end of 1970 where it was known as 'Number Place'. It was published in a magazine called: "Math Puzzles and Logic Problems".

A sudoku is a particular case of what is called a **Euler square** which is an $n\times n$ grid such that each row and column of this grid must be filled with the n distinct numbers without repetitions.

A **magic square** is a 3×3 grid such that the sum of every row, column and diagonal equals 15. One can discuss the maximum number of magic squares that can appear in a unique solution sudoku; for a discussion on this topic see [5].

The resolution of the sudoku can be seen as a problem of coloring the vertices of a plane graph with 81 vertices such that two of them are adjacent if they belong to the same row, column or 3×3 block. If we assign a different color to each of the 9 digits, vertices with the same color cannot be adjacent which means that a digit cannot be twice in each row or each column or each 3×3 block.

Let $x_1,\ldots,x_{81}$ the 81 squares which form the sudoku, arranged from left to right and from top to bottom. Its solution will be $(a_1,\ldots,a_{81})$, where a_i is the number in thesquare associated to the variable x_i. Let S be a sudoku with preassigned data $\{c_i\}_{i\in L}$, for $L \subset \{1,\dots,81\}$. All the necessary information to solve the sudoku is contained in the algebraic set $\mathbb{V}(I+ < \{x_i - c_i\}_{i\in L} >)$. We also consider generalized sudokus. We shall use Groebner basis to find a solution. Some of the results are based on the Masters' Thesis of A. Delgado Latournerie [1].

The minimum number of preassigned data c_i for a unique solution sudoku is 17. This result was proven by McGuire-Tugemann-Civario [3] using computer software. For an interesting reading on the subject, see [4].

The rate of difficulty of a sudoku is ranked using stars. One-star sudokus are very easy to solve while a 5-star sudoku is very hard to solve. There is a famous sudoku, created by the Finnish mathematician Arto Inkala in 2012, whose difficulty is ranked with 11 stars and it is called Everest sudoku. We are not going to discuss here the definition of difficulty for a sudoku.

First we introduce some basic concepts on Algebraic Geometry and on Groebner basis. In **3** we solve sudokus using Groebner basis. In **4** we give SAGE codes for its resolution. In **5, 6** and **7** we study generalizations of sudoku puzzles.

2. BASIC CONCEPTS

Definition 1. *Let R be a commutative ring. Let $I \subset R$ be an ideal of R.*

- *Let $I \neq R$. I is a* **prime ideal** *of R if whenever $ab \in I$, either $a \in I$ or $b \in I$.*
- *Let J be an ideal of R such that $I \subset J$. I is a* **maximal ideal** *if $I = J$ or $J = R$.*
- *The* **radical** *of I is the ideal $\sqrt{I} = \{a \in R : a^n \in I, for\, some\, n \in \mathbb{N}\}$.*
- *I is a* **radical ideal** *if $I = \sqrt{I}$.*

Definition 2. *Let $k[x_1,\dots,x_n]$ be a polynomial ring with coefficients in the field k. Let $\mathbb{A}_k{}^n$ denote the* **affine space** *of dimension n over k. For $J \subset k[x_1,\dots,x_n]$ we define*

$$\mathbb{V}(J) = \{P \in \mathbb{A}_k{}^n : f(P) = 0, \forall f \in J\}.$$

$\mathbb{V}(J)$ *is called* **affine algebraic set.**

Definition 3. *Let $S \subset \mathbb{A}_k{}^n$. We define*

$$\mathbb{I}(S) = \{f \in k[x_1,\dots,x_n] : f(P) = 0, \forall P \in S\}.$$

Properties

- $\mathbb{V}(0) = \mathbb{A}_k{}^n$, $\mathbb{V}(k[x_1,\dots,x_n]) = \emptyset$.
- $\mathbb{I}(\emptyset) = k[x_1,\dots,x_n]$, $\mathbb{I}(\mathbb{A}_k{}^n) = 0$.
- For $J_1 \subset J_2$, $\mathbb{V}(J_2) \subset \mathbb{V}(J_1)$.
- For $S_1 \subset S_2$, $\mathbb{I}(S_2) \subset \mathbb{I}(S_1)$.
- Let $\{S_j\}$, $j \in J$, be a collection of subsets of $\mathbb{A}_k{}^n$. $\sum_{j\in J} \mathbb{I}(S_j) = \mathbb{I}(\cap_{j\in J} S_j)$.
- Let $\{J_i\}$, $i \in I$, be a collection of subsets of $k[x_1,\dots,x_n]$. $\cap_{i\in I}\mathbb{V}(J_i) = \mathbb{V}(\sum_{i\in I} J_i)$.

Definition 4. *We define the* **Zariski topology** *in $\mathbb{A}_k{}^n$ the topology whose closed sets are affine algebraic sets.*

Definition 5. *An affine algebraic set $S \subset \mathbb{A}_k{}^n$ is* **irreducible** *if, whenever $S = S_1 \cup S_2$, for $S_i \subset \mathbb{A}_k{}^n$, $1 \leq i \leq 2$, $S = S_1$ or $S = S_2$. An irreducible affine algebraic set is called an* **algebraic variety**.

Theorem 6. **Hilbert's Nullstellensatz** *Let k be an algebraically closed field. Let $\mathfrak{A}$ be an ideal in $k[x_1,\dots,x_n]$. $\mathbb{I}(\mathbb{V}(\mathfrak{A})) = \sqrt{\mathfrak{A}}$.*

Proof. See [2; Th. 13A]. □

Proposition 7. *Let k be an algebraically closed field. There is a one-to-one correspondence between the ideals of $k[x_1,\dots,x_n]$ and the sets of ${\mathbb{A}_k}^n$ such that radical ideals of $k[x_1,\dots,x_n]$ correspond to affine algebraic sets; the prime ideals of $k[x_1,\dots,x_n]$ correspond to algebraic varieties and the maximal ideals of $k[x_1,\dots,x_n]$ correspond to the points of ${\mathbb{A}_k}^n$.*

Proof. See [2; 1.4, 1.4.4]. □

Proposition 8. *Let $I \subset k[x_1,\dots,x_n]$ be an ideal such that $\mathbb{V}(I)$ is finite. Then,*

- *(a) $|\mathbb{V}(I)| \leq dim_k(\frac{k[x_1,\dots,x_n]}{I})$,*
- *(b) Let k be an algebraically closed field. If I is a radical ideal, $|\mathbb{V}(I)| = dim_k(\frac{k[x_1,\dots,x_n]}{I})$; that is, the number of points of $|\mathbb{V}(I)|$, counted with its multiplicity, is exactly $dim_k(\frac{k[x_1,\dots,x_n]}{I})$.*

Proof. See [1; Prop. 5]. □

Definition 9. *A* **monomial** *in the variables $x_1,\dots,x_n$ is a product of the form $x^\alpha = {x_1}^{\alpha_1},\dots,{x_n}^{\alpha_n}$, with $\alpha = (\alpha_1,\dots,\alpha_n) \in {\mathbb{Z}_{\geq 0}}^n$. An ideal $I \subset k[x_1,\dots,x_n]$ is a* **monomial ideal** *if it admits a system of generators which are monomials.*

Definition 10. *Let $f = \sum_\alpha a_\alpha x^\alpha$ a nonzero polynomial in $k[x_1,\dots,x_n]$. Let $>$ be a monomial ordering.*

- *The* **multidegree** *of f is $multideg(f) = max\{\alpha \in {\mathbb{Z}_{\geq 0}}^n : a_\alpha \neq 0\}$.*
- *The* **leading coefficient** *of f is $LC(f) = a_{multideg(f)}$.*
- *The* **leading monomial** *of f is $LM(f) = x^{multideg(f)}$ with coefficient 1.*
- *The* **leading term** *of f is $LT(f) = a_{multideg(f)} x^{multideg(f)}$*

Definition 11. *Let $I \subset k[x_1,\dots,x_n]$ be a nonzero ideal. $LT(I)$ denotes the set of the leading terms of the elements of I.*

Definition 12. *Let us fix a monomial ordering. A finite subset $G = \{g_1,\dots,g_t\}$ of an ideal $I \subset k[x_1,\dots,x_n]$ is a* **Groebner basis** *if $< LT(g_1),\dots,LT(g_t) >=< LT(I) >$, where LT denotes the leading term as defined in Definition 10.*

Definition 13. *Let $\alpha = (\alpha_1,\dots,\alpha_n)$ and $\beta = (\beta_1,\dots,\beta_n) \in {\mathbb{Z}_{\geq 0}}^n$. We say that $\alpha >_{lex} \beta$ and, thus, that $x^\alpha >_{lex} x^\beta$, if the first nonzero term of the vector $\alpha - \beta$ is posittive.*

Definition 14. *A Groebner basis* **G** *is* **reduced**, *for a polynomial ideal I, if*

- *$LC(f) = 1$, $\forall f \in G$.*
- *For every $f \in G$, no monomial of f is in $< LT(G - \{f\}) >$.*

Lemma 15. *Let $m, n \in \mathbb{N}$, $a_{ij} \in \mathbb{N}$. For each j, $1 \leq j \leq m$, let $\{x_{ij} - a_{ij}\}_{i=1}^n$, be a reduced Groebner basis of the ideal $I_j \subset k[\{x_{ij}\}_{i=1}^n]$. Then $\cup_{j=1}^m(\{x_{ij} - a_{ij}\}_{i=1}^n)$, $a_{ij} \in \mathbb{N}$, $1 \leq i \leq n$, $1 \leq j \leq m$, is a reduced Groebner basis of the ideal $\sum_{j=1}^m I_j \subset k[\{\{x_{ij}\}_{i=1}^n\}_{j=1}^m]$.*

Proof. Let $m, n \in \mathbb{N}$, $a_{ij} \in \mathbb{N}$. By hypothesis, for each j, $1 \leq j \leq m$, let $\{x_{ij} - a_{ij}\}_{i=1}^n$, $a_{ij} \in \mathbb{N}$, be a reduced Groebner basis of the ideal I_j; thus it satisfies the conditions of Definition 14. Since all the monomials are linear, $\cup_{j=1}^m(\{x_{ij} - a_{ij}\}_{i=1}^n)$, $1 \leq i \leq n$, $1 \leq j \leq m$, is a reduced Groebner basis of the ideal $\sum_{j=1}^m I_j \subset k[\{\{x_{ij}\}_{i=1}^n\}_{j=1}^m]$. □

Example 16. *See [1; Ej. 12]. Let* $k = \mathbb{C}$.

Let $I = < x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1 >$.

A Groebner basis for I *with respect to the lexicographical order can be calculated using the mathematical software SAGE using the command* $.grobner\_basis()$

$$R. < x, y, z >= PolynomialRing(CC, 3, order =' lex')$$
$$I = ideal(x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1)$$
$$I.grobner\_basis()$$

We get

$$\{z^2 + y + x - 1, y^2 - y - z^2 + z, 2yz^2 + z^4 - z^2,$$
$$z^6 - 4z^4 + 4z^3 - z^2\} \tag{1}$$

If we just want a Groebner basis obtained from the Buchberger we should use

$$I.grobner\_basis('toy : buchberger')$$

Thus, we obtain

$$\{x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1, y^2 - y - z^2 + z,$$
$$-yz^4 - yz^2 - 2z^4 + 2z^3, -2yz^2 - z^4 + z^2, z^6 - 4z^4 + 4z^3 - z^2\} \tag{2}$$

(1) *and* (2) *generate the same ideal* I.

Definition 17. *Let* f, $g \in k[x_1, \dots, x_n]$ *be two nonzero polynomials. Let* $multideg(f) = \alpha$, $multideg(g) = \beta$. *Let* $\gamma_i = max(\alpha_i, \beta_i)$, $1 \le i \le n$.

- *The monomial* x^γ *is the* **least common multiple** *of* $LM(f)$ *and* $LM(g)$.
- *The* **S-polynomial** *of* f *and* g *is*

$$S(f, g) = \frac{x^\gamma}{LT(f)} f - \frac{x^\gamma}{LT(g)} g.$$

Proposition 18. *Let* $I \subset k[x_1, \dots, x_n]$ *be a nonzero ideal. There exists a Groebner basis of* I *with respect to every monomial order. Moreover, any Groebner basis of* I *is a system of generators of* I.

Proof. See [1; Cor. 2]. □

3. RESOLUTION OF SUDOKUS USING GROEBNER BASIS

Let us consider the sudoku grid

Let us denote by $x_1,\dots,x_{81}$ the 81 squares which form the sudoku, arranged from left to right and from top to bottom. Its solution will be $(a_1,\dots,a_{81})$, where a_i is the number in the square associated to the variable x_i. Let us consider the polynomial ring $\mathbb{C}[x_1,\dots,x_{81}]$.

Definition 19. • *(a) For $1 \le i \le 81$, let $F_i(x_i) = \prod_{k=1}^{9}(x_i - k)$.*

- *(b) For $1 \le i < j \le 81$, let*

$$G_{ij}(x_i,x_j) = \frac{F_i(x_i) - F_j(x_j)}{x_i - x_j} \in \mathbb{Q}[x_i,x_j], \text{ with } i \neq j.$$

Remark 20. *Note that $F_i(x_i)-F_j(x_j)$ is 0 in $\mathbb{V}(x_i-x_j)$; thus (x_i-x_j) is a factor of $F_i(x_i)-F_j(x_j)$ for $i \neq j$. Moreover $G_{ij}(x_i,x_j)$ is not divisible by (x_i-x_j), since, for $x_i = x_j$, $G_{ij}(x_i,x_i) = F_i'(x_i)$ which is not 0 since F_i is not constant. Also, if a_i, a_j are such that $G_{ij}(a_i,a_j) = 0$ and $F_i(a_i) = 0 = F_j(a_j)$ we would have that $a_i \neq a_j$, for, otherwise, $G_{ij}(a_i,a_i) = F_i'(a_i) \neq 0$ because there would be a summand of F_i' which would not be 0 in $a_i \in \{1,\dots,9\}$.*

Notation 21. *Let $E = \{(i,j) \in T\}$*

$(i,j) \in T$ when $1 \le i < j \le 81$ and the ith and jth cells belong to the same row, column or 3×3 block.

We consider the ideal I generated by the polynomials F_i, $1 \le i \le 81$, and G_{ij}, $(i,j) \in E$. Let S be a sudoku with preassigned data $\{c_i\}_{i\in L}$, for $L \subset \{1,\dots,81\}$.

Remark 22. *The ideal $I + <\{x_i - c_i\}_{i\in L}>$ is the ideal generated by $I \cup <\{x_i - c_i\}_{i\in L}>$ and*

$$\mathbb{V}(I + <\{x_i - c_i\}_{i\in L}>) = \mathbb{V}(I) \cap \mathbb{V}(<\{x_i - c_i\}_{i\in L}>).$$

Proposition 23. *The following statements are equivalent:*

- *(1) Let $L \subset \{1,\dots,81\}$. We have $a = (a_1,\dots,a_{81}) \in \mathbb{V}(I + <\{x_i - c_i\}_{i\in L}>$, where c_i are certain constants.*
- *(2) $a_i \in \{1,\dots,9\}$, for $i \in \{1,\dots,81\}$, with $a_i \neq a_j$, for $(i,j) \in E$, and $a_i = c_i$, for all $i \in L$.*

Proof. See [1; Prop. 4].

(1) $\Longrightarrow$ (2) Let $a \in \mathbb{V}(I + <\{x_i - c_i\}_{i\in L}>)$. Then $F_i(a_i) = 0$, $\forall i \in \{1,\dots,81\}$ and $a_i = c_j$, for all $i \in L$. Let us prove that $a_i \neq a_j$, for $(i,j) \in E$. Let us assume that there is a pair $(k,l) \in E$ such that $a_k = b = a_l$. Since $F_k(x_k) = F_l(x_l) + (x_k - x_l)G_{kl}(x_k,x_l)$, we obtain $F_k(x_k) = (x_k - b)G_{kl}(x_k,b)$. If $G_{kl}(b,b) = 0$, we would obtain that b is a zero of F_k of multiplicity at least 2 which is impossible.

(2) $\Longrightarrow$ (1) If for all $i \in \{1,\dots,81\}$ a_i satisfies the first condition of (2) and also $a_i = c_i$, for all $i \in L$, then every F_i, $i \in \{1,\dots,81\}$, would be 0 on them and $x_i - c_i = 0$, for all $i \in L$. If also a_i, $i \in \{1,\dots,81\}$, satisfies the second condition of (2), then G_{ij} would be 0 on $a = (a_1,\dots,a_{81})$. ◻

Remark 24. *Let S be a sudoku with preassigned data $\{c_i\}_{i\in L}$, for $L \subset \{1,\dots,81\}$. Let $I_S = I + <\{x_i - c_i\}_{i\in L}>$ be the ideal associated to the sudoku S. Since $I_S \subset \mathbb{C}[x_1,\dots,x_{81}]$. We have $|\mathbb{V}(I_S)| < 9^{81} < \infty$ and I_S is a radical ideal. By Proposition 8,*

$$|\mathbb{V}(I_S)| = dim_{\mathbb{C}}(\frac{\mathbb{C}[x_1,\dots,x_{81}]}{I_S}).$$

We want to find out when the solution is unique.

Proposition 25. *The following statements are equivalent:*

- *(1) $a = (a_1, \dots, a_{81})$ is the unique solution of the sudoku S.*
- *(2) $I_S =< \{x_1 - a_1, \dots, x_{81} - a_{81}\} >$.*
- *(3) $\{x_1 - a_1, \dots, x_{81} - a_{81}\}$ is a reduced Groebner basis of I_S.*

Proof. See [1; Prop. 6].

(1) $\Longrightarrow$ (2) Since a is a solution $a \in \mathbb{V}(I_S)$. Since a is the unique solution $a = \mathbb{V}(I_S)$ because if there would exist $b \in \mathbb{V}(I_S)$, such that $b \neq a$, b would also be a solution of S which would contradict the unicity of a.

By Theorem 6, $\sqrt{I_S} = \mathbb{I}(\mathbb{V}(I_S))$. Thus, $\sqrt{I_S} = \mathbb{I}(a) =< \{x_1 - a_1, \dots, x_{81} - a_{81}\} >$. In particular, for $i \in \{1, \dots, 81\}$, there exists $m_i \in \mathbb{N}$ such that $(x_i - a_i)^{m_i} \in I_S$. Therefore, $I_S \cap k[x_i] = < (x_i - a_i)^t >$, for some $t \in \mathbb{N}$. But $t = 1$ since the polynomials $F_l(x_l)$ are free of squares. Thus, $I_S =< \{x_1 - a_1, \dots, x_{81} - a_{81}\} >$.

(2) $\Longrightarrow$ (3) We have to see that $\{x_1 - a_1, \dots, x_{81} - a_{81}\}$ is a reduced Groebner basis. Let $G := \{x_1 - a_1, \dots, x_{81} - a_{81}\}$. It is easy to see that G is a Groebner basis since, for each pair (i, j), with $i \neq j$, the remainder of the division of $S(x_i - a_i, x_j - a_j)$ by G is 0. Moreover, for each $i \in \{1, \dots, 81\}$, $LC(x_i - a_i) = 1$ and no monomial of $x_i - a_i$ belongs to $< LT(G - \{x_i - a_i\}) >$, $< LT(G - \{x_i - a_i\}) >=< \{x_j, 1 \leq j \leq 81, i \neq j\} >$. Thus, $\{x_1 - a_1, \dots, x_{81} - a_{81}\}$ is a reduced Groebner basis of I_S.

(3) $\Longrightarrow$ (2) Since $\{x_1 - a_1, \dots, x_{81} - a_{81}\}$ is a reduced Groebner basis of I_S, by Proposition 18, $I_S =< \{x_1 - a_1, \dots, x_{81} - a_{81}\} >$.

(2) $\Longrightarrow$ (1) By Proposition 23 we know that the solutions of a sudoku are the zeroes of the ideal I_S. Thus, if $I_S =< \{x_1 - a_1, \dots, x_{81} - a_{81}\} >$, then $(a_1, \dots, a_{81})$ is the unique solution. □

4. SAGE CODES TO SOLVE A SUDOKU

See also [1, pp. 54–56].

(1) Let us construct E, the polynomials F_i and G_{ij} and the ideal I generated by them.

```
# the set E:
R = Integers(3)
E = [ ]
    for j in[1,.,9] :
    for k in[1,.,9] :
i = (j - 1) * 9 + k
a = R(j)
if a == 0 :
a = 3
b = R(k)
i f b == 0 :
b = 3
# Pairs (i, j) such that x_i and x_j belong to the same row:
    for l in[k + 1,.,9] :
E · append((i, i + l - k)
```

Pairs (i, j) such that x_i and x_j belong to the same column:

for l in$[j+1,.,9]$:

$E\cdot$append$((i,(l-1)*9+k))$

Pairs (i, j) such that x_i and x_j belong to the same 3×3 block:

for m in$[a,.,3]$:

for n in$[1,.,3]$:

if $i<((j-1)+(3-m))*9+k-b+n$:

$E\cdot$append$((i,((j-1)+(3-m))*9+k-b+n)$

print(E)

Polynomials F_i:

F = [ ]

variable = [0]

P(x) = prod([(x - j)for j in [1,.,9]])

for j in$[1,.,81]$:

variable.append(var('x'+ str(j)))

printP (variable[j])

Polynomials G_{ij} :

var('y')

G(x,y) = (P(x) - P(y))/(x- y)

TheGij = [G(variable[a], variable[b]) for a, b in E]

(2) Now, we are going to construct the ideal I_S, generated by the polynomials F_i, $i\in\{1,\ldots,81\}$, and G_{ij}, $(i,j)\in E$, and by the polynomials corresponding to the preassigned values to the sudoku. In a matrix 9×9, the values not preassigned will be written as 0.

Polynomials corresponding to the preassigned values:

def Preassigned(matrix)

polynomial = [ ]

for i in$[0,.,8]$:

for j in$[0,.,8]$:

if matrix[i,j] ! = 0

$k=i*9+(j+1)$

p = variable[k] - matrix[i, j]

polynomial.append(p)

return(polynomial)

T = PolynomialRing(QQ; ['x' + str(j) for j in [1,.,81]])

M = matrix(QQ; matrix[i, j] for i,j in [1,.,9])

L = Preassigned(M)

TheGij = [G(variable[a]; variable[b]) for a, b in E]

I = T.ideal(L + TheGij)

show($I \cdot groebner\_basis$())

$U = I \cdot groebner\_basis$()

N1=U[54:56]

N2=U[63:65]

N3=U[72:74]

N=N1+N2+N3

x='x' + str(j) for j in [1,.,81]

a='a' + str(j) for j in [1,.,81]

def g(x-a)=a

show g(x-a) for x-a in N

Let us consider the following sudoku grid S

		6		4				
	9		8		7			
	8	2				3		
						8	4	
							6	5
1		5						7
4			1	6			9	
		1	4					
	2		7		3			8

Let us calculate the Groebner basis of the ideal I_S
R = PolynomialRing(QQ; ['x' + str(j) for j in [1,..,81]])

M = matrix(QQ; [[0, 0, 6, 0, 4, 0, 0, 0, 0], [0, 9, 0, 8, 0, 7, 0, 0, 0],

[0, 8, 2, 0, 0, 0, 3, 0, 0], [0, 0, 0, 0, 0, 0, 8, 4, 0], [0, 0, 0, 0, 0, 0, 0, 6, 5]

[1, 0, 5, 0, 0, 0, 0, 0, 7], [4, 0, 0, 1, 6, 0, 0, 9, 3], [0, 0, 1, 4, 0, 0, 0, 0, 0],

[0, 2, 0, 7, 0, 3, 0, 0, 8]])

L = Preassigned(M)

TheGij = [G(variable[a]; variable[b]) for a; b in E]

I = R:ideal(L + TheGij) Once L added, the polynomials F_i are redundant.
show($I \cdot groebner\_basis$())

$[x_1 - 3;\ x_2 - 1;\ x_3 - 6;\ x_4 - 2;\ x_5 - 4;\ x_6 - 5;\ x_7 - 7;\ x_8 - 8;\ x_9 - 9;\ x_{10} - 5;$

$x_{11} - 9;\ x_{12} - 4;\ x_{13} - 8;\ x_{14} - 3;\ x_{15} - 7;\ x_{16} - 6;\ x_{17} - 2;\ x_{18} - 1;\ x_{19} - 7;$

$x_{20} - 8;\ x_{21} - 2;\ x_{22} - 9;\ x_{23} - 1;\ x_{24} - 6;\ x_{25} - 3;\ x_{26} - 5;\ x_{27} - 4;\ x_{28} - 9;$

$x_{29} - 6;\ x_{30} - 3;\ x_{31} - 5;\ x_{32} - 7;\ x_{33} - 1;\ x_{34} - 8;\ x_{35} - 4;\ x_{36} - 2;\ x_{37} - 2;$

$x_{38} - 7;\ x_{39} - 8;\ x_{40} - 3;\ x_{41} - 9;\ x_{42} - 4;\ x_{43} - 1;\ x_{44} - 6;\ x_{45} - 5;\ x_{46} - 1;$

$x_{47} - 4;\ x_{48} - 5;\ x_{49} - 6;\ x_{50} - 8;\ x_{51} - 2;\ x_{52} - 9;\ x_{53} - 3;\ x_{54} - 7;\ x_{55} - 4;$

$x_{56} - 5;\ x_{57} - 7;\ x_{58} - 1;\ x_{59} - 6;\ x_{60} - 8;\ x_{61} - 2;\ x_{62} - 9;\ x_{63} - 3;\ x_{64} - 8;$

$x_{65} - 3;\ x_{66} - 1;\ x_{67} - 4;\ x_{68} - 2;\ x_{69} - 9;\ x_{70} - 5;\ x_{71} - 7;\ x_{72} - 6;$

$x_{73}-6;\ x_{74}-2;\ x_{75}-9;\ x_{76}-7;\ x_{77}-5;\ x_{78}-3;\ x_{79}-4;$

$x_{80}-1;\ x_{81}-8]$

The unique solution is:

(3, 1, 6, 2, 4, 5, 7, 8, 9, 5, 9, 4, 8, 3, 7, 6, 2, 1, 7, 8, 2, 9, 1, 6,

3, 5, 4, 9, 6, 3, 5, 7, 1, 8, 4, 2, 2, 7, 8, 3, 9, 4, 1, 6, 5, 1, 4, 5,

6, 8, 2, 9, 3, 7, 4, 5, 7, 1, 6, 8, 2, 9, 3, 8, 3, 1, 4, 2, 9, 5, 7, 6,

6, 2, 9, 7, 5, 3, 4, 1, 8)

Notice that the solution is unique as it is known, by Proposition 8, (b), that

$$|\mathbb{V}(I_S)| = \dim_{\mathbb{C}}(\frac{\mathbb{C}[x_1,\ldots,x_{81}]}{I_S}) = 1.$$

If we solve it using the command “Sudoku( )” of SAGE we obtain

sudoku(M)

S = Sudoku(M)

print(‘The number of solutions of this sudoku is:’)

len(list(S.dlx())) # It computes the number of solutions of this sudoku.

3	*1*	6	*2*	4	*5*	*7*	*8*	*9*
5	9	*4*	8	*3*	7	*6*	*2*	*1*
7	8	2	*9*	*1*	*6*	3	*5*	*4*
9	*6*	*3*	*5*	*7*	*1*	8	4	*2*
2	*7*	*8*	*3*	*9*	*4*	*1*	6	5
1	*4*	5	*6*	*8*	*2*	*9*	*3*	7
4	*5*	*7*	1	6	*8*	*2*	9	3
8	*3*	1	4	*2*	*9*	*5*	*7*	*6*
6	2	*9*	7	*5*	3	*4*	*1*	8

5. GENERALIZATIONS OF SUDOKU PUZZLES

Definition 26. *We call a* **generalized sudoku** *an* $(mn)\times(mn)$ *grid,* $m, n \in \mathbb{N}$, $m, n \geq 2$, *divided into* $m\times n$ *boxes where there are numbers between 1 and* mn *in some of the squares in the grid. To solve the puzzle we have to find the remaining numbers in such a way that every row and every column only contain the digits 1 to* mn *and also every* $m\times n$ *box only contains those digits, without repetitions*

Remark 27.
- *When* $m = n = 2$ *the generalized sudoku is called* **Shidoku** *(see VII).*
- *When* $m = 2$ *and* $n = 3$ *the generalized sudoku is called* **Roku sudoku**.
- *When* $m = 2$ *and* $n = 4$ *the generalized sudoku is called* **Hachi sudoku**.
- *When* $m = 3$ *and* $n = 4$ *there is a type of generalized sudoku which uses the digits from 1 to 9 and the letters A, B and C instead of the digits 10, 11 and 12. This type is called* **Juuni sudoku**.
- *When* $m = n = 4$ *there is a type of generalized sudoku which uses the digits from 1 to 9 and the letters A, B, C, D, E, F and G instead of the digits 10, 11, 12, 13, 14, 15 and 16. This type is called* **Supersudoku**.

Let $r = (mn)^2$. Let $x_1, \ldots, x_r$ the r squares which form the sudoku, arranged from left to right and from top to bottom. Its solution will be $(a_1, \ldots, a_r)$, where a_i is the number in the square associated to the variable x_i. Let S be a generalized sudoku with preassigned data $\{c_i\}_{i\in L}$, for $L \subset \{1, \ldots, r\}$. All the necessary information to solve the sudoku is contained in the algebraic set $\mathbb{V}(I+ < \{x_i - c_i\}_{i\in L} >)$.

Definition 28.
- *(a) For $1 \le i \le r$, let $F_i(x_i) = \prod_{k=1}^{mn}(x_i - k)$, $k \in \{1, \ldots, mn\}$, $i \in \{1, \ldots, r\}$.*
- *(b) For $1 \le i < j \le r$, let*

$$G_{ij}(x_i, x_j) = \frac{F_i(x_i) - F_j(x_j)}{x_i - x_j} \in \mathbb{Q}[x_i, x_j], \textit{with } i \ne j.$$

Remark 29. *Note that $F_i(x_i) - F_j(x_j)$ is 0 in $\mathbb{V}(x_i - x_j)$; thus $(x_i - x_j)$ is a factor of $F_i(x_i) - F_j(x_j)$ for $i \ne j$. Moreover $G_{ij}(x_i, x_j)$ is not divisible by $(x_i - x_j)$, since, for $x_i = x_j$, $G_{ij}(x_i, x_i) = F_i'(x_i)$ which is not 0 since F_i is not constant. Also, if a_i, a_j are such that $G_{ij}(a_i, a_j) = 0$ and $F_i(a_i) = 0 = F_j(a_j)$ we would have that $a_i \ne a_j$, for, otherwise, $G_{ij}(a_i, a_i) = F_i'(a_i) \ne 0$ because there would be a summand of F_i' which would not be 0 in $a_i \in \{1, \ldots, mn\}$.*

Notation 30. *Let $r = (mn)^2$. Let $E = \{(i, j) \in T\}$.*

$(i, j) \in T$ when $1 \le i < j \le r$ and the ith and jth cells belong to the same row, column or $m \times n$ block.

We consider the ideal I generated by the polynomials F_i, $1 \le i \le r$, and G_{ij}, $(i, j) \in E$. Let S be a sudoku with preassigned data $\{c_i\}_{i\in L}$, for $L \subset \{1, \ldots, r\}$, $r = (mn)^2$.

Remark 31. *The ideal $I+ < \{x_i - c_i\}_{i\in L} >$ is the ideal generated by $I \cup < \{x_i - c_i\}_{i\in L} >$ and*

$$\mathbb{V}(I+ < \{x_i - c_i\}_{i\in L} >) = \mathbb{V}(I) \cap \mathbb{V}(< \{x_i - c_i\}_{i\in L} >)$$

Proposition 32. *Let $r = (mn)^2$. The following statements are equivalent:*

- *(1) Let $L \subset \{1, \ldots, r\}$. We have $a = (a_1, \ldots, a_r) \in \mathbb{V}(I+ < \{x_i - c_i\}_{i\in L} >$, where c_i are certain constants.*
- *(2) $a_i \in \{1, \ldots, mn\}$, for $i \in \{1, \ldots, r\}$, with $a_i \ne a_j$, for $(i, j) \in E$, and $a_i = c_i$, for all $i \in L$.*

Proof. See Prop. 23.

(1) $\Longrightarrow$ (2) Let $a \in \mathbb{V}(I+ < \{x_i - c_i\}_{i\in L} >)$. Then $F_i(a_i) = 0$, $\forall i \in \{1, \ldots, r\}$ and $a_i = c_j$, for all $i \in L$. Let us prove that $a_i \ne a_j$, for $(i, j) \in E$. Let us assume that there is a pair $(k, l) \in E$ such that $a_k = b = a_l$. Since $F_k(x_k) = F_l(x_l) + (x_k - x_l)G_{kl}(x_k, x_l)$, we obtain $F_k(x_k) = (x_k - b)G_{kl}(x_k, b)$. If $G_{kl}(b, b) = 0$, we would obtain that b is a zero of F_k of multiplicity at least 2 which is impossible.

(2) $\Longrightarrow$ (1) If for all $i \in \{1, \ldots, r\}$ a_i satisfies the first condition of (2) and also $a_i = c_i$, for all $i \in L$, then every F_i, $i \in \{1, \ldots, r\}$, would be 0 on them and $x_i - c_i = 0$, for all $i \in L$. If also a_i, $i \in \{1, \ldots, r\}$, satisfies the second condition of (2), then G_{ij} would be 0 on $a = (a_1, \ldots, a_r)$. □

Remark 33. *Let S be a sudoku with preassigned data $\{c_i\}_{i\in L}$, for $L \subset \{1, \ldots, r\}$. Let $I_S = I+ < \{x_i - c_i\}_{i\in L} >$ be the ideal associated to the sudoku S. Since $I_S \subset \mathbb{C}[x_1, \ldots, x_r]$. We have $|\mathbb{V}(I_S)| < (mn)^r < \infty$ and I_S is a radical ideal. By Proposition 8,*

$$|\mathbb{V}(I_S)| = dim_{\mathbb{C}}(\frac{\mathbb{C}[x_1, \ldots, x_r]}{I_S}).$$

We want to find out when the solution is unique.

Proposition 34. *Let $r = (mn)^2$. The following statements are equivalent:*

- *(1) $a = (a_1, \dots, a_r)$ is the unique solution of the sudoku S.*
- *(2) $I_S =< \{x_1 - a_1, \dots, x_r - a_r\} >$.*
- *(3) $\{x_1 - a_1, \dots, x_r - a_r\}$ is a reduced Groebner basis of I_S.*

Proof. See Prop. 25.

(1) $\Longrightarrow$ (2) Since a is a solution $a \in \mathbb{V}(I_S)$. Since a is the unique solution $a = \mathbb{V}(I_S)$ because if there would exist $b \in \mathbb{V}(I_S)$, such that $b \neq a$, b would also be a solution of S which would contradict the unicity of a.

By Theorem 6, $\sqrt{I_S} = \mathbb{I}(\mathbb{V}(I_S))$. Thus, $\sqrt{I_S} = \mathbb{I}(a) =< \{x_1 - a_1, \dots, x_r - a_r\} >$. In particular, for $i \in \{1, \dots, mn\}$, there exists $m_i \in \mathbb{N}$ such that $(x_i - a_i)^{m_i} \in I_S$. Therefore, $I_S \cap k[x_i] = < (x_i - a_i)^t >$, for some $t \in \mathbb{N}$. But $t = 1$ since the polynomials $F_l(x_l)$ are free of squares. Thus, $I_S =< \{x_1 - a_1, \dots, x_r - a_r\} >$.

(2) $\Longrightarrow$ (3) We have to see that $\{x_1 - a_1, \dots, x_r - a_r\}$ is a reduced Groebner basis. Let $G := \{x_1 - a_1, \dots, x_r - a_r\}$. It is easy to see that G is a Groebner basis since, for each pair (i, j), with $i \neq j$, the remainder of the division of $S(x_i - a_i, x_j - a_j)$ by G is 0. Moreover, for each $i \in \{1, \dots, r\}$, $LC(x_i - a_i) = 1$ and no monomial of $x_i - a_i$ belongs to $< LT(G - \{x_i - a_i\}) >$, $< LT(G - \{x_i - a_i\}) >=< \{x_j, 1 \leq j \leq r, i \neq j\} >$. Thus, $\{x_1 - a_1, \dots, x_r - a_r\}$ is a reduced Groebner basis of I_S.

(3) $\Longrightarrow$ (2) Since $\{x_1 - a_1, \dots, x_r - a_r\}$ is a reduced Groebner basis of I_S, by Proposition 18, $I_S =< \{x_1 - a_1, \dots, x_r - a_r\} >$.

(2) $\Longrightarrow$ (1) By Proposition 32 we know that the solutions of a sudoku are the zeroes of the ideal I_S. Thus, if $I_S =< \{x_1 - a_1, \dots, x_r - a_r\} >$, then $(a_1, \dots, a_r)$ is the unique solution. □

6. SAGE CODES TO SOLVE A GENERALIZED SUDOKU

Let S be a generalized sudoku which is an $(mn) \times (mn)$ grid, $m, n \in \mathbb{N}$, $m, n \geq 2$, $m \leq n$.

(1) Let us construct E, the polynomials F_i and G_{ij} and the ideal I generated by them.

```
# the set E:
R1 = Integers(m)
t = mn
E = [ ]
    for j in[1,., t] :
    for k in[1,., t] :
i = (j − 1) ∗ t + k
a = R1(j)
if a == 0 :
a = m
R2 = Integers(n)
b = R2(k)
i f b == 0 :
b = m
```

Pairs (i, j) such that x_i and x_j belong to the same row:

for l in$[k+1,.,t]$:

$E \cdot$ append$((i, i+l-k)$

Pairs (i, j) such that x_i and x_j belong to the same column:

for l in$[j+1,.,t]$:

$E \cdot$ append$((i,(l-1)*t+k))$

Pairs (i, j) such that x_i and x_j belong to the same $m \times n$ block:

for u in$[a,.,m]$:

for w in$[1,.,m]$:

if $i < ((j-1)+(m-u))*t+k-b+w$:

$E \cdot$ append$((i,((j-1)+(m-u))*t+k-b+w)$

print(E)

Polynomials F_i:

$r = t*t$

F = [ ]

variable = [0]

P(x) = prod([(x - j)for j in [1,.,t]])

for j in$[1,.,r]$:

variable.append(var('x'+ str(j)))

printP (variable[j])

Polynomials G_{ij} :

var('y')

G(x,y) = (P(x) - P(y))/(x- y)

TheGij = [G(variable[a], variable[b]) for a, b in E]

(2) Now, we are going to construct the ideal I_S, generated by the polynomials F_i, $i \in \{1,\dots,mn\}$, and G_{ij}, $(i, j) \in E$, and by the polynomials corresponding to the preassigned values to the sudoku. In a matrix $(mn) \times (mn)$, the values not preassigned will be written as 0.

Polynomials corresponding to the preassigned values:

def Preassigned(matrix)

polynomial = [ ]

$t = mn$

$r = t*t$

for i in$[0,.,t-1]$:

for j in$[0,.,t-1]$:

if matrix[i,j] ! = 0

$k = i*t+(j+1)$

```
p = variable[k] - matrix[i, j]
polynomial.append(p)
return(polynomial)
T = PolynomialRing(QQ; ['x' + str(j) for j in [1,..,r]])
M = matrix(QQ; matrix[i, j] for i,j in [1,..,t])
L = Preassigned(M)
TheGij = [G(variable[a]; variable[b]) for a, b in E]
I = T.ideal(L + TheGij)
show(I·groebner_basis())
```

7. OTHER TYPES OF SUDOKU PUZZLES

Shidoku: It is a 4×4 grid divided into four 2×2 boxes where there are numbers between 1 and 4 in some of the squares in the grid (for example the bolded black below). To solve the puzzle we have to find the remaining numbers in such a way that every row and every column only contain the digits 1 to 4 and also every 2×2 box only contains those digits, without repetitions (for example the black below).

3	2	**4**	1
4	**1**	2	3
2	3	1	**4**
1	4	**3**	**2**

Let $x_1, \ldots, x_{16}$ the 16 squares which form the shidoku S_1, arranged from left to right and from top to bottom. Its solution will be $(a_1, \ldots, a_{16})$, where a_i is the number in the square associated to the variable x_i, with preassigned data $\{c_i\}_{i \in L_1}$, for $L_1 \subset \{1, \ldots, 16\}$.

Remark 35. *Let S be a shidoku with preassigned data* $\{c_i\}_{i \in L}$, *for* $L \subset \{1, \ldots, 16\}$. *Let* $I_S = I + < \{x_i - c_i\}_{i \in L} >$ *be the ideal associated to the shidoku S, where I is the ideal generated by the polynomials* F_i, $1 \leq i \leq 16$, *and* G_{ij}, $(i, j) \in E$, *defined in a similar way to the ones in Definition 19 and Notation 21.* $E = \{(i, j) : for\, 1 \leq i < j \leq 16$ *the ith and jth cells belong to the same row, column or* $2 \times 2\, block\}$.

Proposition 36. *The following statements are equivalent:*

- *(1)* $a = (a_1, \ldots, a_{16})$ *is the unique solution of the shidoku S.*
- *(2)* $I_S = < \{x_1 - a_1, \ldots, x_{16} - a_{16}\}. >$
- *(3)* $\{x_1 - a_1, \ldots, x_{16} - a_{16}\}$ *is a reduced Groebner basis of* I_S.

Proof. Similar to the one of Prop. 25. □

Several sudokus sharing 3×3 blocks

Let us denote the sudokus by S_j, $1 \leq j \leq m$, arranged in such a way that some of them share a 3×3 block. Let S denote such a configuration with associated ideal I_S. Let I_{S_j} be the ideal associated to the sudoku S_j, $1 \leq j \leq m$, $1 \leq a_{ij} \leq 9$. $I_{S_j} \subset k[\{x_{ij}\}_{i=1}^{81}]$. For each j, $1 \leq j \leq m$, let $\{x_{ij}-a_{ij}\}_{i=1}^{81}$, be a reduced Groebner basis of the ideal I_{S_j}. Then $\cup_{j=1}^{m}(\{x_{ij}-a_{ij}\}_{i=1}^{81})$, is a reduced Groebner basis of the ideal $\sum_{j=1}^{m} I_{S_j} \subset k[\{\{x_{ij}\}_{i=1}^{81}\}_{j=1}^{m}]$, by Lemma 15. Notice that some of the x_{ij} are equal to some x_{kl} and that some of the a_{ij} are equal to some a_{kl} since some 3×3 blocks may be shared. Let us assume that, in S, we have g distinct x_{ij} that we shall denote by $y_1,\dots,y_g$ and where b_i is the digit in $\{1,\dots,9\}$ associated to the variable y_i.

Proposition 37. *The following statements are equivalent:*

- *(1)* $b=(b_1,\dots,b_g)$ *is the unique solution of* S.
- *(2)* $I_S=<\{y_1-b_1,\dots,y_g-b_g\}>$.
- *(3)* $\{y_1-b_1,\dots,y_g-b_{16}\}$ *is a reduced Groebner basis of* I_S.

Proof. Similar to the one of Prop. 25. □

Three sudokus joined in diagonal

We have three sudokus S_i, $1\le i\le 3$, arranged in diagonal in such a way that S_i and S_{i+1}, $1\le i\le 2$, share a corner 3×3 block.

Let $x_1,\dots,x_{81}$ the 81 squares which form the sudoku S_1, arranged from left to right and from top to bottom. Its solution will be $(a_1,\dots,a_{81})$, where a_i is the number in the square associated to the variable x_i, with preassigned data $\{c_i\}_{i\in L_1}$, for $L_1\subset\{1,\dots,81\}$. Similarly, let $y_1,\dots,y_{81}$ the 81 squares which form the sudoku S_2, arranged from left to right and from top to bottom. Its solution will be $(b_1,\dots,b_{81})$, where b_i is the number in the square associated to the variable y_i, with preassigned data $\{d_i\}_{i\in L_2}$, for $L_2\subset\{1,\dots,81\}$. S_1 and S_2 share a 3×3 block; thus, $x_{54+l}=y_{6+l}$, $l=9m+s$, $1\le s\le 3$, $0\le m\le 2$.

Let $z_1,\dots,z_{81}$ the 81 squares which form the sudoku S_3, arranged from left to right and from top to bottom. Its solution will be $(e_1,\dots,e_{81})$, where e_i is the number in the square associated to the variable z_i, with preassigned data $\{g_i\}_{i\in L_3}$, for $L_3\subset\{1,\dots,81\}$. S_3 and S_2 share a 3×3 block; thus, $y_{54+l}=z_{6+l}$, $l=9m+s$, $1\le s\le 3$, $0\le m\le 2$.

We consider the polynomial ring $R=\mathbb{C}[x_1,\dots,x_{81},y_1,\dots,y_{81},z_1,\dots,z_{81}]$, with $x_{54+l}=y_{6+l}$, $y_{54+l}=z_{6+l}$, $l=9m+s$, $1\le s\le 3$, $0\le m\le 2$. Let $R_1=\mathbb{C}[x_1,\dots,x_{81}]$, $R_2=\mathbb{C}[y_1,\dots,y_{81}]$, $R_3=\mathbb{C}[z_1,\dots,z_{81}]$ be subrings of R.

Let I_{S_i} be the ideals associated to the sudoku S_i, $1\le i\le 3$; $I_{S_i}\subset R_i$, $1\le i\le 3$.

Remark 38. *The unique solution of the three sudokus is given by* $\cap_{i=1}^{3}\mathbb{V}(I_{S_i})=\mathbb{V}(\sum_{i=1}^{3} I_{S_i})$. *If each one of the sudokus has unique solution,* $\{x_i-a_i\}_{i=1}^{81}$, $\{y_i-b_i\}_{i=1}^{81}$, $\{z_i-e_i\}_{i=1}^{81}$, *are, respectively, reduced Groebner basis of the ideals* I_{S_i}, $1\le i\le 3$, *then, by Lemma 15,* $\cup_{j=1}^{3}(\{x_i-a_i,y_i-b_i,z_i-e_i\}_{i=1}^{81})$, a_i, b_i, $e_i\in\{1,\dots,9\}$, $1\le i\le 81$, *is a reduced Groebner basis of the ideal* $\sum_{i=1}^{3} I_{S_i}\subset k[\{x_i,y_i,z_i\}_{i=1}^{81}]$, *where* $x_{(54+l)}=y_{(6+l)}$, $a_{(54+l)}=b_{(6+l)}$, *and* $y_{(54+l)}=z_{(6+l)}$, $b_{(54+l)}=e_{(6+l)}$, *for* $l=9m+s$, $1\le s\le 3$, $0\le m\le 2$; *so it is the unique solution for the three of them, by Proposition 37.*

It can be generalized to several sudokus S_i, $1\le i\le k$, *arranged in diagonal in such a way that* S_i *and* S_{i+1}, $1\le i\le k-1$, *share a corner* 3×3 *block.*

The preassigned data $\{c_i\}_{i\in L_1}$, $\{d_i\}_{i\in L_2}$ and $\{g_i\}_{i\in L_3}$ are written in black in the example below (Figure 1).

Samurai Sudoku: It is composed of five sudokus joined in the shape of X. There is a central sudoku with four sudokus at the edges. Each shares a 3×3 block with the central sudoku.

Let us denote the five sudokus by S_j, $1\le j\le 5$, arranged in such a way that S_3 is the sudoku at the center sharing a 3×3 block with S_j, $1\le j\le 4$, $j\ne 3$, the other four sudokus located at the corners of the central one.

												1	5	4	2	3	8	6	7	9
												2	7	6	1	5	9	3	4	8
												3	8	9	4	6	7	5	1	2
												4	1	2	3	8	6	7	9	5
												6	3	5	7	9	2	4	8	1
												8	9	7	5	1	4	2	3	6
						8	6	2	1	5	7	9	4	3	6	2	1	8	5	7
						9	7	4	2	3	8	5	6	1	8	7	3	9	2	4
						3	1	5	6	4	9	7	2	8	9	4	5	1	6	3
						7	8	3	9	1	2	6	5	4						
						6	5	1	8	7	4	3	9	2						
						4	2	9	3	6	5	8	1	7						
1	4	8	5	7	9	2	3	6	7	9	1	4	8	5						
2	6	7	1	3	4	5	9	8	4	2	3	1	7	6						
3	5	9	2	6	8	1	4	7	5	8	6	2	3	9						
4	7	1	3	2	6	8	5	9												
5	8	2	4	9	1	6	7	3												
6	9	3	7	8	5	4	2	1												
7	1	4	6	5	3	9	8	2												
8	2	5	9	1	7	3	6	4												
9	3	6	8	4	2	7	1	5												

Figure 1

Remark 39. *The unique solution of the five sudokus is given by* $\cap_{j=1}^{5}\mathbb{V}(I_{S_j}) = \mathbb{V}(\sum_{i=1}^{5} I_{S_j})$. *If each one of the sudokus has unique solution, for each* j, $1 \le j \le 5$, *let* $\{x_{ij} - a_{ij}\}_{i=1}^{81}$, *be a reduced Groebner basis of the ideal* $I_j \subset k[\{x_{ij}\}_{i=1}^{81}]$, *then, by Lemma 15,* $\cup_{j=1}^{5}(\{x_{ij} - a_{ij}\}_{i=1}^{81})$, $a_{ij} \in \{1,\ldots,9\}$, $1 \le i \le 81$, $1 \le j \le 5$, *is a reduced Groebner basis of the ideal* $\sum_{j=1}^{5} I_j \subset k[\{\{x_{ij}\}_{i=1}^{81}\}_{j=1}^{5}]$, *where* $x_{(54+l)2} = x_{(6+l)3}$ *and* $x_{(54+l)3} = x_{(6+l)5}$ *and also* $x_{(54+t)1} = x_{l3}$, $x_{(54+t)3} = x_{l4}$, *for* $l = 9m + s$, $1 \le s \le 3$, $t = 9m + v$, $7 \le v \le 9$, $0 \le m \le 2$; *thus it is the unique solution for the five of them, by Proposition 37.*

The preassigned data are written in black in the example below (Figure 2).

Sohei sudoku (Figure 3)

It is composed of four sudokus forming a cross + arranged in such a way that each sudoku shares a 3×3 block with each one of the two adjacent ones.

Let us denote the four sudokus by S_j, $1 \le j \le 4$, arranged in such a way that S_1 shares a 3×3 block with S_2 and S_4, S_2 shares a 3×3 block with S_3 and S_1, S_3 shares a 3×3 block with S_2 and S_4.

Remark 40. *The unique solution of the four sudokus is given by* $\cap_{j=1}^{4}\mathbb{V}(I_{S_j}) = \mathbb{V}(\sum_{i=1}^{4} I_{S_j})$. *If each one of the sudokus has unique solution, for each* j, $1 \le j \le 4$, *let* $\{x_{ij} - a_{ij}\}_{i=1}^{81}$, *be a reduced Groebner basis of the ideal* $I_j \subset k[\{x_{ij}\}_{i=1}^{81}]$, *then, by Lemma 15,* $\cup_{j=1}^{4}(\{x_{ij} - a_{ij}\}_{i=1}^{81})$, $a_{ij} \in \{1,\ldots,9\}$, $1 \le i \le 81$, $1 \le j \le 4$, *is a reduced Groebner basis of the ideal* $\sum_{j=1}^{4} I_j \subset k[\{\{x_{ij}\}_{i=1}^{81}\}_{j=1}^{4}]$, *where* $x_{(54+l)1} = x_{(6+l)4}$ *and* $x_{(54+l)2} = x_{(6+l)3}$, *for* $l = 9m + s$, $1 \le s \le 3$, *and also* $x_{(54+t)1} = x_{l2}$, $x_{(54+t)4} = x_{l3}$, *for* $t = 9m + v$, $7 \le v \le 9$, $0 \le m \le 2$; *thus it is the unique solution for the four of them, by Proposition 37.*

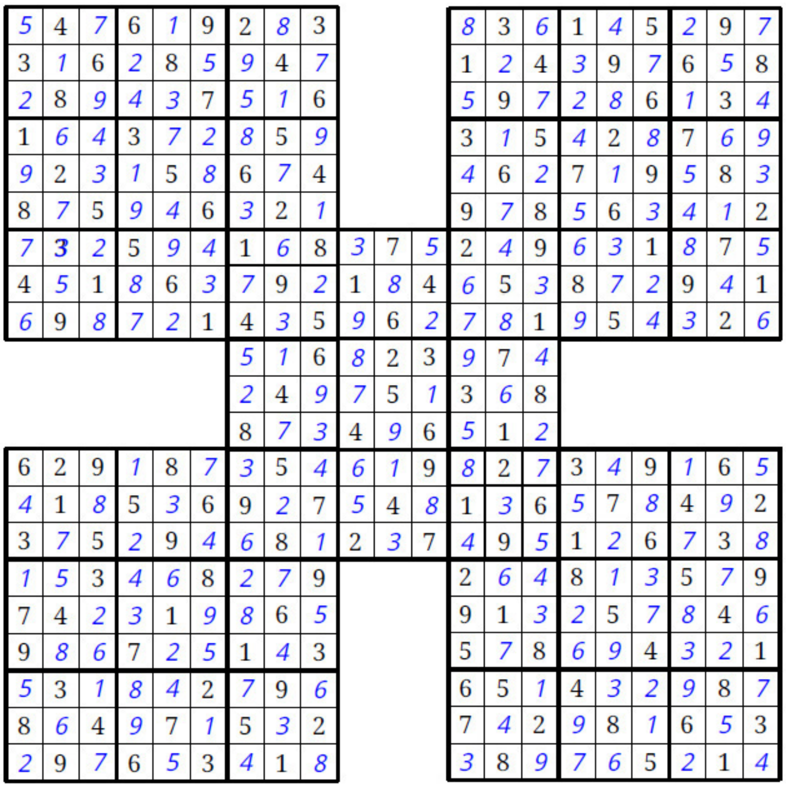

Figure 2

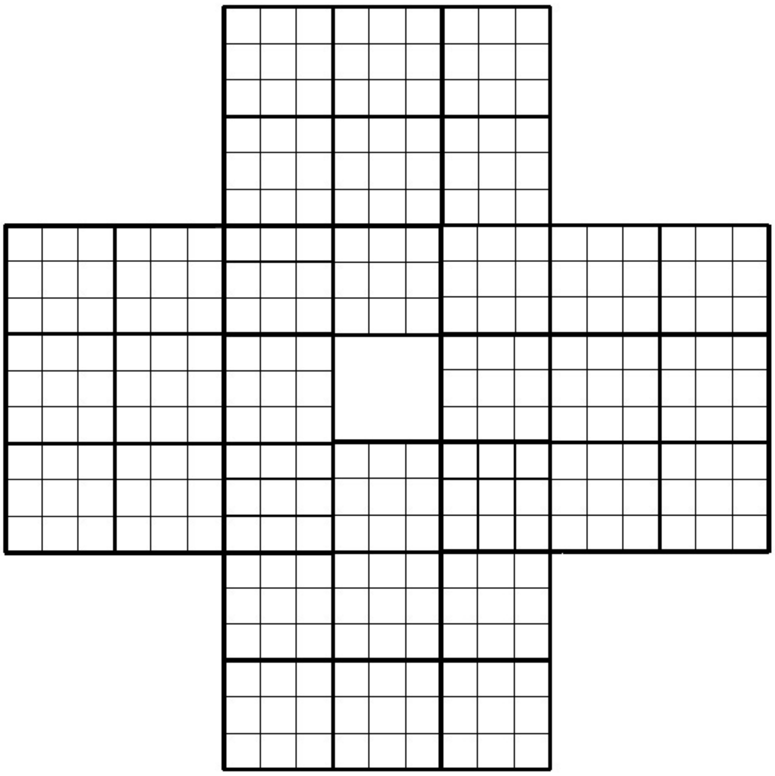

Figure 3

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РЕШЕНИЕ СУДОКУ С ПОМОЩЬЮ БАЗИСОВ ГРЕБНЕРА

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Аннотация

Мы изучаем решение судоку и обобщенного судоку, используя технику базисов Грёбнера. Пусть $x_1,\dots,x_{81}$ переменные, связанные с 81 квадратами, которые образует головоломку судоку и линейно упорядочены сначала по строкам, затем по столбцам. Решение судоку есть набор чисел $(a_1,\dots,a_{81})$, где a_i число в квадрате, ассоциированном с переменной x_i.

Пусть также S — судоку с предварительно заполненными данными $\{c_i\}_{i\in L}$ для $L\subset\{1,\dots,81\}$. Вся необходимая информация для решения такого судоку содержится в алгебраическом множестве $\mathbb{V}(I+<\{x_i-c_i\}_{i\in L}>)$. Мы используем технику базисов Грёбнера для поиска такого решения и приводим соответсвующий код в системе компьютерной алгебры SAGE для программы, решающей эту задачу.

Ключевые слова: *судоку, обобщённое судоку, базис Грёбнера, алгебраическое многообразие.*

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