Computer tools in education, 2017
№ 4: 14–28
http://ipo.spb.ru/journal

ON COMPUTER MODELING OF FINITE-GENERATED FREE PROJECTIVE PLANES

Gogin N. D., Mylläri A. A.[1]

[1]St. George's University, Grenada, West Indies

Abstract

This paper treats computer modeling of the process of constructing free projective planes — more precisely, to algorithmically finding their successive incidence matrices; and also to considering some numerical characteristics of these matrices. Matrix and bilinear forms approaches are used to study the growth rate of the number of new elements (points, lines) during step-by-step process of constructing projective plane starting with the Hall Π^4 configuration. It appears that the number of new elements grows asymptotically as a double exponent (linear on $log(log)$ scale.) Rough estimate from above also gives double exponential growth rate.

Keywords: *free projective planes, finite geometries, combinatorial design.*

Citation: N. D. Gogin & A. A. Mylläri, "On Computer Modeling of Finite-generated Free Projective Planes", *Computer tools in education*, no. 4, pp. 14–28, 2017.

1. INTRODUCTION

W. W. Sawyer in his Prelude to Mathematics [1] writes: "Projective geometry is one of the most beautiful parts of elementary mathematics.

For the professional mathematician it is undoubtedly an essential part of one's education. One does not need to go very far with it; the value of a detailed study of it is doubtful, except for the specialist. But the basic patterns of projective geometry can be traced in many other branches of mathematics; they serve to guide and to unify."

The subject of this paper is free and finite projective planes, part of the vast area of modern combinatorics, called the theory of combinatorial designs. Combinatorial design theory is the part of combinatorial mathematics that deals with the existence, construction and properties of systems of finite sets whose arrangements satisfy generalized concepts of balance and/or symmetry [2]. Applications of combinatorial design theory can be found in many areas including finite geometry (finite affine and projective planes, Möbius or inversive planes, etc.), tournament scheduling, experimental design, lotteries, mathematical biology, algorithm design and analysis, networking, finite groups theory, and cryptography. We address interested readers to the previously cited article in Wikipedia and references therein. Combinatorial designs have a long history: for example, the magic square of order three, the so-called Lo Shu Square, dates at least to 650 BC; the oldest image of this square was found on a tortoiseshell dated 2200 BC (according to legend the Chinese Emperor Yu observed the magic square

$$\begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix}$$

on the back of a divine tortoise [11].) Combinatorial design methods evolved along with the general growth of combinatorics from the 18th century, for example, from the studies of Latin squares and the famous "36 officers problem", which goes back to Leonard Euler (1782) [11]. Today, one can see many people solving Sudoku puzzles — actually, they are solving a classic combinatorial design problem.

Classical subjects of combinatorial design theory include balanced incomplete block designs (BIBDs), symmetric BIBDs, Hadamard matrices and Hadamard designs, difference sets. Other combinatorial designs are related to or have been developed from the study of these fundamental ones.

Let us give for the sake of completeness, the definition of BIBD (balanced incomplete block design), or (b, v, r, k, λ)-configuration [11]. Let X be a finite set of v elements. A balanced incomplete block design (or simply block design) is a collection B of b subsets (blocks) of X, such that every block has the same number k of elements, each pair of distinct elements appear together in the same number λ of blocks, where $k < v - 1$, $\lambda > 0$, and any element of X is contained (replicated) in the same number r of blocks.

It follows immediately from the definition that $r(k-1) = \lambda(v-1)$ and $bk = vr$.

A symmetric balanced incomplete block design (SBIBD), (v, k, λ)-configuration is a BIBD in which the number of elements equals the number of blocks ($v = b$). They are the single most important and well studied subclass of BIBDs.

A finite projective plane of order n is SBIBD with parameters $v = n^2 + n + 1$, $k = n + 1$, $\lambda = 1$.

The theory of combinatorial designs in general and of finite geometries in particular abounds with a mass of unsolved problems that are difficult to be investigated even with modern methods of combinatorial mathematics. This also applies to the theory of projective planes (see for example the "Prime-power hypothesis for the orders of the finite projective planes" below). In particular, no sufficiently developed general theory of construction and the structure of finite projective planes has been created to date.

In view of this, it seems quite natural that in an effort to create such a theory, mathematicians turned to already known analogous constructions usually called "free objects" of the theory in question. In our case, we are talking about "free projective planes", which being infinite themselves, can shed light on problems associated with finite projective planes.

Of course, the study of free projective planes is also of great interest by itself.

Free projective planes were first introduced by M. Hall in his fundamental paper [3] where he considered their basic properties. Since then, these planes have become the subject of constant interest of mathematicians studying abstract algebraic structures, group theory and their representations, and so on [4, 5, 7**?** , 8]. There are also good surveys which one can use to get acquainted with the basic concepts and achievements of the modern theory of combinatorial geometries, for example, [6, 10–12]. As a general introduction to the projective geometry, one can use e.g. [13–15].

This paper is devoted to computer modeling of the process of constructing free projective planes — more precisely, to algorithmically finding their successive incidence matrices; and to considering some numerical characteristics of these matrices.

Remarks about notations: If A is a (non-empty) matrix then $dim1(A)$ (resp. $dim2(A)$) is a number of its rows (resp. columns); $[A]_{i,j}$ means its element at the entry (i, j); A_i (resp. A^j)

means i-th row (resp. j-th column); $diag(A)$ for a square matrix A means column-vector of its diagonal elements; $Total[A]$ is a sum of all elements in A. Moreover, we treat binomial coefficient $\binom{x}{2}$ and differential operators (derivatives, Laplace operator) as listable functions.

$\eta_{i,j}$ is a column-vector with "1"-s only in two different positions i and j and all the rest components equal to "0"-s.

As a rule we do not show the matrix format explicitly unless it is not clear from context.

E denotes identity matrix; J is a square constant matrix of (only) "1"-s; $J^* = J - E$; {} denotes empty matrix; $\langle\ ,\ \rangle$ means Euclidean scalar product; for a matrix A and real α we define a product $\alpha \bullet A$ as follows:

$$\alpha \bullet A = \begin{cases} \alpha A, & \text{if } \alpha \neq 0 \\ \{\}, & \text{if } \alpha = 0. \end{cases}$$

$A \circ B$ denotes the element-wise (Hadamard) product of matrices with the identical formats.

If A and B are matrices having appropriate formats then $A| \cup B$(resp. $\underline{A} \cup B$) denotes a concatenation of A and B from the right (resp. from below) providing $A| \cup \{\} = \underline{A} \cup \{\} = A$.

2. PRELIMINARIES

In this section we mostly follow the terminology and definitions of [6].

Definition 1. A configuration (or a partial plane [1]) is a pair $\Pi = (P, L)$ where P is (nonempty) set of points and L is a family of subsets of P called lines under the condition that the following axiom is valid:

C1: Any two different points are incident with no more than one line.

Axiom C1 implies

C2: Any two different lines are incident with no more than one point in common.

As a rule in this paper we shall be interested only in the case of finite sets P.

Examples 1.

1. Desargues' configuration directly related to the Desargues' theorem (a classic example of the projective theorem, completely independent of measurement) is well-known (see, e.g. [1], [6]): Mark a point O, draw the three lines OA, OB, OC. Points A, B, and C can be anywhere on these lines. Also choose any three points A', B', C', A' on OA, B' on OB, C' on OC. Join AB and $A'B'$. These two lines intersect in point F. In the same way, AC and $A'C'$ intersect in point E, BC and $B'C'$ intersect in point D.

Desargues' theorem for the usual real projective plane claims: points D, E, and F lie on a straight line (see Fig. 1).

Desargues' configuration consists of 10 lines, each incident to 3 points, from the other side, there are 10 points, each incident to 3 lines. It has a strong symmetry: any of these 10 points could be marked as O, there always will be a way (actually, several ways) to mark other points so that the statement of the theorem remains true. There are 120 different ways of putting in the letters on the picture without any changes in the printed statement being necessary [1].

2. Another classic example of the projective theorem, completely independent of measurement is Pappus theorem. Pappus configuration we get by taking two lines and choosing three points A, B, C on one line and three points A', B', and C' on another line (points should be different from the intersection point of this two lines.) Connect A with B' and C'; connect B with A' and C'; connect C with A' and B'. Let us denote intersection of lines AB' and $A'B$ by D, intersection of lines AC' and CA' by E, intersection of lines CA' and $C'A$ by F.

Pappus' theorem: Points D, E, and F are collinear (see Fig. 2.)

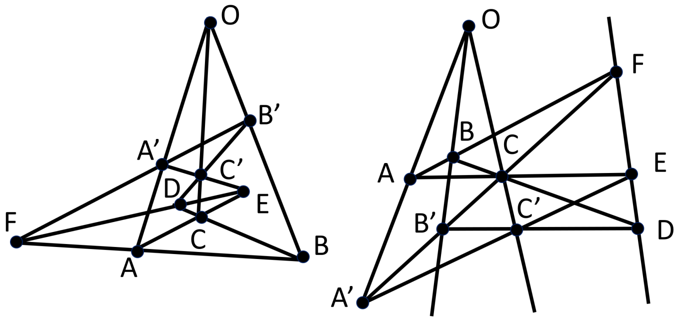


Figure 1. Two examples of the Desargues' configuration. Points D, E, and F lie on a straight line

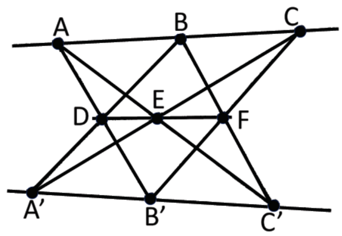


Figure 2. Pappus' configuration. Points D, E, and F lie on a straight line

Pappus' configuration consists of 9 lines, each incident to 3 points, from the other side, there are 9 points, each incident to 3 lines (cf. [1, 6]).

3. If in Definition 1 $L = \emptyset$ and $|P| = m$, $m > 0$ is an integer, then we have a pure m-points configuration.

4. If L consists of all pairs $\{a, b\}$, $a, b \in P$, $a \neq b$ then $\Pi = (P, L)$ is a full graph on m vertices.

5. Let $\Pi^m = (P, L)$, $m \geq 4$ be a configuration with $|P| = m$ and only one line λ, (i.e. $L = \{\lambda\}$) where $|\lambda| = m-2$. This means that all points besides two of them lie on the (unique) line λ. These configurations are called standard [10] or Hall configurations and were first introduced by M. Hall in his fundamental paper [3], p. 237.

Definition 2. Configuration $\Pi = (P, L)$ is called a projective plane, if in axioms C1 and C2 the words "...with no more than one..." are changed by "... exactly one...", i.e. in $\Pi = (P, L)$ the following axioms are valid:

P1: Any two different points are incident to exactly one line;

P2: Any two different lines are incident to exactly one point in common;

and in addition the axiom

P3: There exist 4 different points such that no three of them are collinear; in order to exclude some degenerate configurations (cf. [10]).

The following simple statements can be easily proved for a finite projective plane [4]:

A) Every line is incident to exactly $n+1$ points;

B) Every point is incident to exactly $n+1$ lines;

C) $|P| = |L| = N = n^2 + n + 1$.

The number n is called the order of the finite projective plane.

Example 2. Fano plane: for $n = 2$ we obtain an example of a "smallest" (nondegenerate) projective plane, called the Fano plane. This plane contains $7 = 2^2 + 2 + 1$ points and 7 lines, each line contains $3 = 2 + 1$ points and through each point pass 3 lines (Fig. 3).

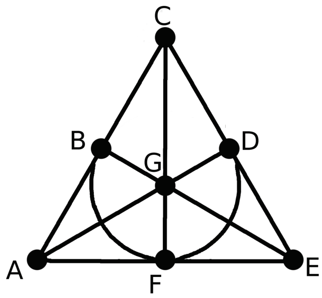


Figure 3. Fano plane — finite projective plane of order two

"Prime-power hypothesis for the orders of the finite projective planes" claims that always $n = p^{\mu}$ for some prime p. To date, this hypothesis remains unproved.

Definition 3. If $\Pi = (P, L)$ is a finite configuration with $|P| = m$ and $|L| = l$, $l > 0$ then the incident matrix of Π is defined as $l \times m$ 0-1-matrix $A = (a_{i,j})$ where

$$a_{i,j} = \begin{cases} 1, & \text{if point } j \text{ is incident with line } i \\ 0, & \text{if point } j \text{ and the line } i \text{ are not incident} \end{cases} \tag{1}$$

in some chosen (and fixed) numerations of sets P and L.

Example 3.

1. Incident matrix of Desargues' configuration (with proper numbering of points $O, A, B, C, A', B', C', D, E, F$) is

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

We leave as an exercise for the reader to find corresponding numberings for Fig. 1.

2. Incident matrix of Pappus configuration with ordering points $A, B, C, A', B', C', D, E, F$ (Fig. 2) is

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

3. Incident matrix of the Fano plane (with proper numbering of points A, B, C, D, E, F, G) is cyclic:

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Again, we leave as an exercise for the reader to find corresponding numbering for Fig. 3.

General properties of the incident matrices are as follows:

a. The i-th row A_i of incident matrix indicates all points incident to the i-th line and

$$\begin{aligned} Total[A_i] &= \sum_{j=1}^{n} a_{i,j} = \sum_{j=1}^{n} a_{i,j}^2 \\ &= \langle A_i, A_i \rangle = \text{(number of points on the } i\text{-th line)} \end{aligned} \tag{2}$$

whereas for $i \neq k$ the scalar product $\langle A_i, A_k \rangle$ is 0 or 1 according to axiom C2.

b. Dually, the j−th column A^j of incident matrix shows all lines incident to the j-th point and

$$\begin{aligned} Total[A^j] &= \sum_{i=1}^{l} a_{i,j} = \sum_{i=1}^{l} a_{i,j}^2 \\ &= \langle A^j, A^j \rangle = \text{(number of lines incedent to the } j\text{-th point)} \end{aligned} \tag{3}$$

whereas for $j \neq k$ the scalar product $\langle A^j, A^j \rangle$ is 0 or 1 according to the axiom C1.

c. So, the i-th diagonal element of the product AA^T equals (number of points on the i-th line), whereas the elements outside the diagonal are 0 or 1. Of course, mutatis mutandis this is valid also for $A^T A$. Obviously

$$Tr(AA^T) = Tr(A^T A) = Total(A) \tag{4}$$

d. If all the outside-diagonal elements in AA^T (resp., $A^T A$) are equal to 1, we say that configuration is line-wise ample (resp. point-wise ample).

Clearly, if $\Pi = (P, L)$ is a projective plane of order n then it is both point-wise ample and line-wise ample and its incident matrix is a square $N \times N$ 0-1-matrix such that

$$AA^T = A^T A = nE + J \tag{5}$$

(cf. for example, [4]).

Example 4. These properties can easily be checked with matricies from the Example 3. For example, for the incident matrix of the Fano plane ($n = 2$) we have

$$AA^T = \begin{pmatrix} 3 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 3 \end{pmatrix},$$

while for the incident matrix of the Pappus configuration we have

$$AA^T = \begin{pmatrix} 3 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 3 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 & 0 & 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 3 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 2 & 1 & 3 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 3 \end{pmatrix}.$$

3. FREE PROJECTIVE PLANE GENERATED BY CONFIGURATION

Let $\Pi_0 = (P_0, L_0)$ be some (initial) configuration. The free projective plane generated by Π_0 is defined by the following process:
1. Let $\Pi_1 = (P_1, L_1)$ be a new configuration where $L_1 = L_0$ and $P_1 = P_0 \cup \nu P_0$

$$\nu P_0 = \{(a)(b) | a, b \in L_0, a \text{ and } b \text{ are not incident in } \Pi_0\} \tag{6}$$

i.e. every pair of non-incident lines defines a new point named $(a)(b)$ which is "intersection" of lines a and b. Evidently Π_1 is line-wise ample.
2. Let $\Pi_2 = (P_2, L_2)$ be a new configuration where $P_2 = P_1$ and $L_2 = L_1 \cup \nu L_1$

$$\nu L_1 = \{(a)(b) | a, b \in P_1, a \text{ and } b \text{ are not incident in } \Pi_1\} \tag{7}$$

i.e. every pair of non-incident points a and b defines a new line named $(a)(b)$ which "connects" points a and b. Evidently Π_2 is point-wise ample.

Iterating this construction we get a sequence (finite or infinite) of configurations $\{\Pi_0, \Pi_1, \Pi_2, \Pi_3, \Pi_4, \Pi_5, \ldots, \Pi_r, \ldots\}$ in which for r even we add new points to Π_r, as in item 1 and for r odd we add new lines to Π_r as in item 2 and get next configuration Π_{r+1}, $r \geq 0$.

Proposition 1 (see [6]). If Π_0 contains 4 different points no three of which are collinear then $\Pi = fr(\Pi_0) = \bigcup_{k=0}^{\infty} \Pi_k$ is a projective plane.

This plane is said to be the free projective plane generated by Π_0.

Remarks:

1. If an initial configuration Π_0 is finite and has isolated ("empty") point(s) (resp. "empty lines") then after the first (resp. "second") step of the above algorithm such point(s) (resp. "lines") will vanish, so in order for the computer realization of the algorithm to be implemented correctly, we must always require that the initial incident matrix (and hence all the next) does not contain zero-columns (resp. "zero-rows").

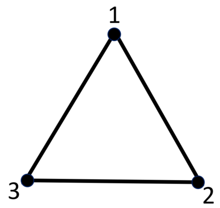
2. The construction of "names" for new points/lines in the above definition gives rise to attempts to consider free projective planes as commutative but not associative universal algebras **[?]**.

Example 5.

1. If Π_0 is a projective plane then evidently $fr(\Pi_0) = \Pi_0$.

2. If $|\Pi_0| = 3$ and $|L_0| = 0$ then $fr(\Pi_0)$ is called a "projective plane of order $n = 1$" (see Definition 2, p.1) and it is a plane over the field of one element (Fig. 4, left). This plane is referred to as "degenerated" because Axiom P3 evidently is not valid for it. Its incident matrix is cyclic.

The following theorem of M. Hall (see [3]) explains the importance of Hall configuration Π^4:



$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Figure 4. Projective plane of order $n = 1$ (left) and its incident matrix (right).

1) Let Π_0 is any non-degenerate configuration but not a projective plane. Then $fr(\Pi_0)$ contains $fr(\Pi^4)$ as a subplane. Moreover, such plane is never desarguesian.

2) A $fr(\Pi^m)$, $m \geq 4$ contains $fr(\Pi^{m+1})$.

Everywhere in what follows we deal only with the Hall configuration Π^4, i.e. $fr(\Pi^4) = \{\Pi^4_r\}_{r=0,1,2,\dots}$, that is "free equivalent"(see [3]) to a pure configuration on 4 points, i.e., a full graph with 4 vertices.

4. MATRIX APPROACH

According to what was said at the end of previous section we begin with configuration $\Pi_0 = \Pi^4$ (which is zero-step, $s = 0$, of our algorithm) with incident matrix

$$A_0 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

which corresponds to the configuration 4 from Example 1 with $m = 4$. This configuration (tetrahedron) is shown below on Fig. 5 (left).

Evidently here $\dim 1(A_0) = \Lambda_0 = 6$, $\dim 2(A_0) = \mathrm{P}_0 = 4$.

Since

$$A_0 A_0^T = \begin{pmatrix} 2 & 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 1 & 2 \end{pmatrix}, \quad A_0^T A_0 = \begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix}$$

this configuration is point-wise ample (any two different points are incident), but is not line-wise ample because exactly 3 pairs of lines, namely 1,4, 2,5 and 3,6, have no points in common.

According to item 1 of the general constructing of $fr(\Pi_0)$ at the next step $s = 1$ we must add to Π_0 $\nu\mathrm{P}_0 = 3$ new points, namely (1)(4), (2)(5) and (3)(6) (see Fig. 5 (center)), that means that we must concatenate (from the right) to A_0 three new columns numbered respectively 5, 6, 7, whereas the number of new lines $\nu A_0 = 0$.

So, here $\dim 1(A_1) = \Lambda_1 = \Lambda_0 = 6$, $\dim 2(A_1) = \mathrm{P}_0 + \nu\mathrm{P}_0 = 4 + 3 = 7$ and the matrix of the next configuration Π_1 (see Fig. 5 (right)) is

$$A_1 = \left(\begin{array}{cccc|ccc} 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

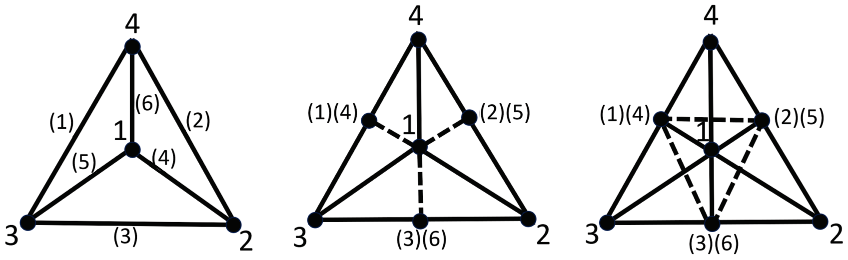


Figure 5. Initial configuration $\Pi_0 = \Pi^4$ (left) and two steps of the algorithm: adding new points (center) and new lines (right)

Note that positions of "1"-s in the concatenated columns are exactly 1 and 4, 2 and 5, and 3 and 6.

Going over to the next step $s = 2$ we find that

$$A_1 A_1^T = \begin{pmatrix} 3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 3 \end{pmatrix},$$

$$A_1^T A_1 = \begin{pmatrix} 3 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

So, here $\dim 1(A_2) = \Lambda + 2 = \Lambda_1 + \nu\Lambda_1 = 6 + 3 = 9$, $\dim 2(A_2) = \mathrm{P}_2 = \mathrm{P}_1 + \nu\mathrm{P}_1 = 7 + 0 = 7$ and the matrix of the next configuration (see Fig. 5 (center)) is

$$A_2 = \left[\begin{array}{ccccccc} 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

Now it is not difficult to describe the general case for any step $s > 0$:

a1) If $s \equiv 1 \mod 2$ we add new points

$$\begin{aligned} \nu P_{s-1} &= \text{(number of non-incident lines at step } s-1) \\ &= \frac{1}{2}\text{(number of "0"-s in } A_{s-1}A_{s-1}^T) \\ &= \binom{\Lambda_{s-1}}{2} - Total\left[\binom{diag(A_{s-1}^T A_{s-1})}{2}\right] \\ &= \binom{\Lambda_{s-1}}{2} - Total\left[\binom{A_{s-1}^T A_{s-1}}{2}\right] \end{aligned} \quad (8)$$

whereas clearly $\nu A_{s-1} = 0$.

Proof of (8): The first and second equalities are evident.

Furthermore, $Total\left[\binom{diag(A_{s-1}^T A_{s-1})}{2}\right] = Total\left[\binom{A_{s-1}^T A_{s-1}}{2}\right]$ because $\binom{1}{2} = 0$. At last, $\binom{\Lambda_{s-1}}{2}$ is equal to the all pairs of different lines at step $s-1$, whereas $Total\left[\binom{diag(A_{s-1}^T A_{s-1})}{2}\right]$ is equal to such pairs of lines which are already incident at this step (see item c of general properties of the incident matrices, p. 2).

For example, for $s = 1$ we get $\nu P_0 = \binom{6}{2} - 4 \cdot \binom{3}{2} = 3$, since $\Lambda_0 = 6$, $diag(A_0^T A_0) = (3333)$.

In other words, here we get the $\Lambda_s \times P_s$-matrix A_s, where $\Lambda_s = \Lambda_{s-1}$, $P_s = P_{s-1} + \nu P_{s-1}$, by concatenating from the right to A_{s-1} one by one new νP_{s-1} columns.

So, in this case we get a formula (we remind that $0 \bullet a = \{\}$):

$$A_s = A_{s-1} | \bigcup_{2 \le i \le A_{s-1} i \le j \le A_{s-1}} \left((1 - [A_{s-1}A_{s-1}^T]_{i,j}) \bullet \eta_{i,j}\right) \quad (9)$$

Dually,

a2) If $s \equiv 0 \mod 2$ we add new lines

$$\begin{aligned} \nu \Lambda_{s-1} &= \text{(number of non-incident points at step } s-1 \\ &= \frac{1}{2}\text{(number of "0"-s in } A_{s-1}^T A_{s-1}) \\ &= \binom{P_{s-1}}{2} - Total\left[\binom{diag(A_{s-1}A_{s-1}^T)}{2}\right] \\ &= \binom{P_{s-1}}{2} - Total\left[\binom{A_{s-1}A_{s-1}^T}{2}\right] \end{aligned} \quad (10)$$

whereas clearly $\nu P_{s-1} = 0$.

For example, for $s = 2$ we get $\nu \Lambda_1 = \binom{7}{2} - 6 \cdot \binom{3}{2}$, since $P_1 = 7$, $diag(A_1 A_1^T) = (333333)$. So, in this case we get a formula:

$$A_s = \frac{A_{s-1}}{} \bigcup_{2 \le i \le A_{s-1} i \le j \le A_{s-1}} \left((1 - [A_{s-1}^T A_{s-1}]_{i,j}) \bullet \eta_{i,j}^T\right). \quad (11)$$

Formulas (9) and (11) give rise to the first variant of our algorithms.

First four steps are illustrated on Fig. 6.

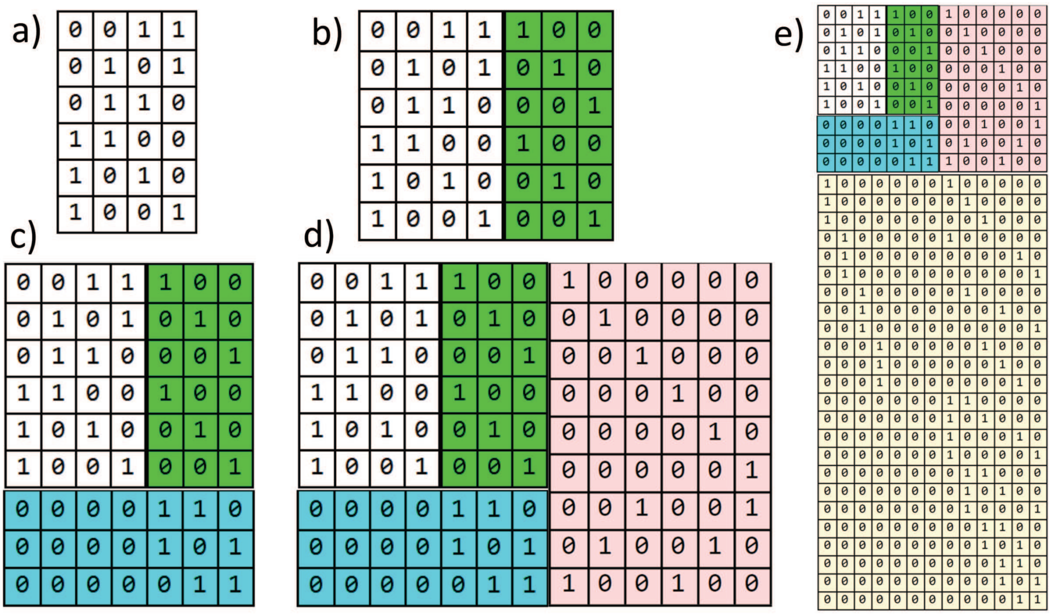


Figure 6. Incident matrix of initial configuration (a) and four first steps of the algorithm: three new points added (b), three new lines added (c), 6 new points added (d), 24 new lines added (e)

5. BILINEAR FORMS APPROACH

Let $\pi = \{p_i\}_{i=1}^{\infty}$ and $\lambda = \{l_i\}_{i=1}^{\infty}$ be two sets of independent variables for points and lines respectively.

For any step $s \geq 0$ we introduce a bilinear form $F_s = F_s(\pi, \lambda) = \pi^T A_s \lambda$ where A_s is an incident matrix constructed on step s (see Sec. 3) and π and λ are initial segments of the infinite sequences of variables π and λ having appropriate lengths. For example, for $s = 0$ we have $\pi = \{p_i\}_{i=1}^{4}$, $\lambda = \{l_i\}_{i=1}^{6}$ and

$$\begin{aligned} F_0 &= l_4 p_1 + l_5 p_1 + l_6 p_1 + l_2 p_2 + l_3 p_2 + l_4 p_2 + l_1 p_3 + l_3 p_3 + l_5 p_3 + l_1 p_4 \\ &\quad + l_2 p_4 + i_6 p_4 \\ &= l_1(p_3 + p_4) + l_2(p_2 + p_4) + l_3(p_2 + p_3) + l_4(p_1 + p_2) + l_5(p_1 + p_3) \\ &\quad + l_6(p_1 + p_4) \\ &= p_1(l_4 + l_5 + l_6) + p_2(l_2 + l_3 + l_4) + p_3(l_1 + l_3 + l_5) + p_4(l_1 + l_2 + l_6). \end{aligned} \tag{12}$$

Now it is clear that also in general case $Coefficient[F_s, l_i] = \frac{\partial F_s}{\partial l_i}$ is a linear form in π representing the i−th row of A_s; $Coefficient[F_s, p_j] = \frac{\partial F_s}{\partial p_j}$ is a linear form in λ representing the j−th column of A_s.

Also it is clear that two lines, l_i and l_k with $1 \leq i, k \leq \Lambda_s$, $i \neq k$ are not incident iff. the linear forms $\frac{\partial F_s}{\partial l_i}$ and $\frac{\partial F_s}{\partial l_k}$ have no variables in common that implies that in this case the Laplace operator in π

$$\Delta_\pi \left(\frac{\partial F_s}{\partial l_i} \cdot \frac{\partial F_s}{\partial l_k} \right) = \sum_{p \in \pi} \frac{\partial^2}{\partial p^2} \left(\frac{\partial F_s}{\partial l_i} \cdot \frac{\partial F_s}{\partial l_k} \right) = 0 \tag{13}$$

and otherwise

$$\Delta_\pi \left(\frac{\partial F_s}{\partial l_i} \cdot \frac{\partial F_s}{\partial l_k} \right) = 2. \tag{14}$$

It's clear that if $i = k$ then

$$\lambda_\pi\left(\left(\frac{\partial F_s}{\partial l_i}\right)^2\right) = 2\cdot\ \text{(number of points on } i-\text{th line)} = 2\left[diag(A_s A_s^T)\right]_i \qquad (15)$$

For example,

$$\Delta_\pi\left(\frac{\partial F_0}{\partial l_1}\cdot\frac{\partial F_0}{\partial l_4}\right) = \sum_{r=1}^{4}\frac{\partial^2}{\partial p_r^2}\left((p_3+p_4)(p_1+p_2)\right) = 0,$$

whereas

$$\Delta_\pi\left(\frac{\partial F_0}{\partial l_1}\cdot\frac{\partial F_0}{\partial l_2}\right) = \sum_{r=1}^{4}\frac{\partial^2}{\partial p_r^2}\left((p_3+p_4)(p_2+p_4)\right) = 2,$$

and

$$\Delta_\pi\left(\frac{\partial F_0}{\partial l_1}\right)^2 = \sum_{r=1}^{4}\frac{\partial^2}{\partial p_r^2}\left((p_3+p_4)^2\right) = 2\cdot 2 = 4.$$

Obviously that formulas dual to (13), (14) and (15) also are valid mutatis mutandis.

Using formulas (13), (14), (15) and their duals it is easy to verify matrices equalities

$$\frac{1}{2}\Delta_\pi\left(\left(\frac{\partial F_s}{\partial \lambda}\right)^{\otimes 2}\right) = A_s A_s^T,\ \frac{1}{2}\Delta_\lambda\left(\left(\frac{\partial F_s}{\partial \pi}\right)^{\otimes 2}\right) = A_s^T A_s, \qquad (16)$$

where $\frac{\partial F_s}{\partial \lambda} = grad_\lambda(f_s)$, $\frac{\partial F_s}{\partial \pi} = grad_\pi(f_s)$, the Laplace operators are supposed to be listable and $^{\otimes 2}$ means tensor square.

Now we are going to write the recurrent formulas from step $s-1$ to step s:

Formulas (8) and (10) may be written in terms of bilinear forms as follows:

$$\nu P_{s-1} = \binom{\Lambda_{s-1}}{2} - Total\left[Binomial\left[\frac{1}{2}\Delta_\pi\left(\frac{\partial F_{s-1}}{\partial \lambda}\right)^2, 2\right]\right], \qquad (17)$$

$$\nu \Lambda_{s-1} = \binom{P_{s-1}}{2} - Total\left[Binomial\left[\frac{1}{2}\Delta_\lambda\left(\frac{\partial F_{s-1}}{\partial \pi}\right)^2, 2\right]\right]. \qquad (18)$$

As to a recurrent relation between forms F_{s-1} and F_s here we have

$$F_s = F_{s-1} + \nu F_{s-1},\quad s \geq 1. \qquad (19)$$

For brevity of writing formulas for we use the reduced Laplace matrices $\overline{\Delta_\pi}(F_s) = J - J^* \circ \frac{1}{2}\Delta_\pi\left(\left(\frac{\partial F_s}{\partial \lambda}\right)^{\otimes 2}\right)$ and $\overline{\Delta_\lambda}(F_s) = J - J^* \circ \frac{1}{2}\Delta_\lambda\left(\left(\frac{\partial F_s}{\partial \pi}\right)^{\otimes 2}\right)$. For example, if $s = 0$ then

$$\frac{1}{2}\Delta_\pi\left(\left(\frac{\partial F_0}{\partial \lambda}\right)^{\otimes 2}\right) = \begin{pmatrix} 2 & 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 1 & 2 \end{pmatrix}$$

and

$$\overline{\Delta_\pi} F(x_0) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

i.e. in $\overline{\Delta_\pi}F(x_0)$ all non-zero elements become "0"-s and all zeros become "1"-s.

Then it is not difficult to check that

for odd step, $s \equiv 1 \bmod 2$,

$$\nu F_{s-1} = \sum_{1\le i\le \Lambda_{s-1}, i\le j\le \Lambda_{s-1}} (l_i + l_j) p_{\sigma(i,j)} \left\lfloor \overline{\Delta_\pi} \right\rfloor_{i,j}, \text{ where } \sigma(i,j) = P_{s-1} + \sum_{\alpha\le i,\beta\le j} \lfloor \overline{\Delta_\pi} \rfloor_{i,j} \tag{20}$$

for even step, $s \equiv 0 \bmod 2$, $s > 0$

$$\nu F_{s-1} = \sum_{1\le i\le P_{s-1}, i\le j\le P_{s-1}} (p_i + p_j) l_{\sigma(i,j)} \left\lfloor \overline{\Delta_\lambda} \right\rfloor_{i,j}, \text{ where } \sigma(i,j) = \Lambda_{s-1} + \sum_{\alpha\le i,\beta\le j} \lfloor \overline{\Delta_\pi} \rfloor_{i,j}. \tag{21}$$

For example, if $s = 1$ then $\nu F_0 = (l_1 + l_4)p_5 + (l_2 + l_5)p_6 + (l_5 + l_6)p_7$ and $F_1 = F_0 + \nu F_0 = l_1(p_3 + p_4) + l_2(p_2 + p_4) + l_3(p_2 + p_3) + l_4(p_1 + p_2) + l_5(p_1 + p_4) + (([_1 + l_4)p_5 + (l_2 + l_5)p_6 + (l_3 + l_6)p_7)$.

Formulas (19), (20), (21) are exact analogs of those (9) and (11) but the "exotic" oncatenations of matrices are changed by usual polynomial additions.

These formulas also give rise to alternative algorithm for recursive construction of $fr(\Pi^4)$.

6. IMPLEMENTATION

As was said above we used matrix and bilinear forms approaches.

The first difficulty in programming was caused by the requirement to avoid zero-columns/rows in incident matrices as well as "fictitious" variables in bilinear forms. This difficulty is surmounted with special procedures for numeration of new constructed columns/rows of matrices and new variables of bilinear forms.

A more serious obstacle is the (above-mentioned) fact of the very fast growth of matrices' formats. Though those are very sparse 0-1-matrices, the programming tools for such matrices provided by Mathematica proved insufficient for our purposes, so computer memory resources became exhausted quickly.

As a result, we managed to calculate only 7 members of the sequence $u_n = \nu \mathrm{P}_n + \nu \Lambda_n$, $n \ge 0$ (note that one of the two summand in "u_n" is always equal to 0):

3, 3, 6, 24, 282, 37233, 684792168, ...

It is easy to check empirically that this sequence grows asymptotically as a double exponent of n (Fig. 7).

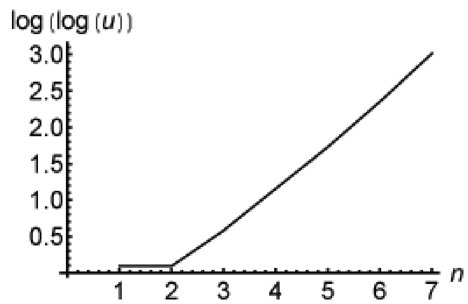


Figure 7. Number of elements grows as double exponent (linear on log(log) scale

Though we failed so far to find a general formula for the number of new elements on each step, we can find the (rather rough) upper bound using (4.1) and (4.3): ignoring second terms we immediately get

$$\nu \mathrm{P}_{s-1} \le \binom{\Lambda_{s-1}}{2}, \quad \nu \Lambda_{s-1} \le \binom{\mathrm{P}_{s-1}}{2}.$$

Assuming $\nu \mathrm{P}_{s-1} = \binom{\Lambda_{s-1}}{2}$ and $\nu \Lambda_{s-1} = \binom{\mathrm{P}_{s-1}}{2}$ we have

$\Lambda_s = \Lambda_{s-1}$, $\mathrm{P}_s = \mathrm{P}_{s-1} + \nu \mathrm{P}_{s-1}$ for s even,

and

$\mathrm{P}_s = \mathrm{P}_{s-1}$, $\Lambda_s = \Lambda_{s-1} + \nu \Lambda_{s-1}$ for s odd. See upper line on Fig. 8.

We can improve this upper bound by taking into account that all diagonal elements in (4.1) and (4.3) always are ≥ 3. We have

$$\nu \mathrm{P}_{s-1} \leq \binom{\Lambda_{s-1}}{2} - 3\mathrm{P}_{s-1}, \quad \nu \Lambda_{s-1} \leq \binom{\mathrm{P}_{s-1}}{2} - 3\Lambda_{s-1}.$$

In both cases we get double exponential growth. These two lines together with our result are shown in Fig. 8.

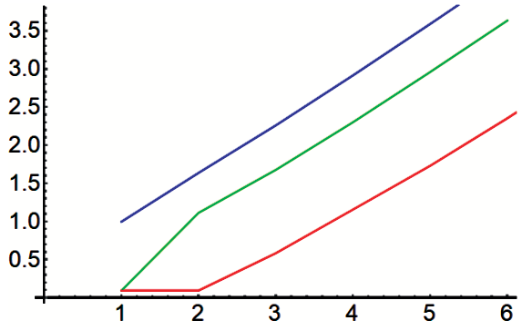


Figure 8. Number of elements (lower line) and two upper bounds

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Поступила в редакцию 04.07.2017, окончательный вариант — 28.07.2017.

Компьютерные инструменты в образовании, 2017
№ 4: 14–28
УДК: 512.56 + 514.146
http://ipo.spb.ru/journal

КОМПЬЮТЕРНОЕ МОДЕЛИРОВАНИЕ КОНЕЧНО-ПОРОЖДЕННЫХ СВОБОДНЫХ ПРОЕКТИВНЫХ ПЛОСКОСТЕЙ

Гогин Н. Д., Милляри А. А.[1]

[1]Университет Св. Георгия, Гренада, Вест-Индия

Аннотация

Работа посвящена компьютерному моделированию процесса построения свободных проективных плоскостей, или более точно, алгоритмическому нахождению их последовательных матриц инцидентности. Рассматриваются также некоторые целочисленные характеристики этих матриц. Матричный метод, а также подход, использующий билинейные формы, применяются для изучения темпов роста числа новых элементов (точек, линий) в процессе поэтапного построения проективной плоскости, начиная с конфигурации М. Холла Π^4. Число новых элементов растет асимптотически как двойная экспонента (линейно по $\log(\log)$ шкале.) Оценка сверху также дает двойной экспоненциальный рост.

Ключевые слова: *свободные проективные плоскости, конечные геометрии, комбинаторные схемы.*

Цитирование: Gogin N. D., Mylläri A. A. On Computer Modeling of Finite-generated Free Projective Planes // Компьютерные инструменты в образовании. 2017. № 4. С. 14–28.

Received 04.07.2017, the final version — 28.07.2017.

Гогин Никита Дмитриевич, кандидат физико-математических наук, доцент, ngiri@list.ru
Мюлляри Александр Альбертович, кандидат физико-математических наук, профессор, Школа Искусств и Наук, университет Сент Джорджес, Гренада, Вест-Индия, amyllari@sgu.edu

Nikita D. Gogin,
PhD, docent,
ngiri@list.ru

Aleksandr A. Mylläri,
PhD, Professor, Department of Computers and Technology, School of Arts and Sciences, St. George's University, Grenada, West Indies,
amyllari@sgu.edu