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AUTOMATED ALGEBRAIC CALCULATION OF INTERACTIVELY CONSTRUCTED GEOMETRIC FIGURES — A DIDACTIC ANALYSIS

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Abstract

Ordinary interactive dynamic geometry systems (IDGS) cannot generally show, how a measurable property of an interactively constructed geometric figure depends on the determining variables of that figure. With a certain method of “Automated Deduction in Geometry” (ADG), implemented in Geometry Expressions (abbreviated GX, homepage: http://www.geometryexpressions.com/), it is now possible to calculate such a dependency as an algebraic expression consisting of the determining variables. We explain this facility by selected examples, which represent the usage of this tool and evaluate its application in classroom.

Keywords: *planimetry, automated calculation, interactive dynamic geometry systems.*

In the past decade significant systems for the learning and teaching of geometry with professional interface user-software have been published. These are the systems such as: Cabri 3D (prototype for interactive dynamic 3D-geometry systems), Cinderella (an interactive dynamic geometry system with a script language for far reaching extensions by programming), GeoGebra (an interactive dynamic geometry system compatible with its modules: spreadsheet, computer algebra and 3D-geometry) and Geometry Expressions (a constraint-based interactive dynamic geometry system — a prototype for automated algebraic calculation of interactively constructed geometric figures and with constructions beyond the ruler and compass). The content of this contribution to learning and instruction of geometry for upper secondary education in mathematics is a didactic analysis of Geometry Expressions’ facility of automated algebraic calculation, which is opening up a new link between plane geometry and algebra.

1. INTRODUCTION

Ordinary interactive dynamic geometry systems (IDGS) cannot generally show, how a measurable property of an interactively constructed geometric figure depends on the determining variables of that figure. With a certain method of “Automated Deduction in Geometry” (ADG), implemented in Geometry Expressions (abbreviated GX, homepage: http://www.geometryexpressions.com/), it is now possible calculating such a dependency as an algebraic expression consisting of the determining variables.

We exemplify this facility by the following example: In Fig. 1 the diagonals of an interactively constructed trapezium intersect in point E; line parallel to base AB through E intersects AD in F and BC in G. A geometric novice guesses that E is midpoint of FG and verifies that by corresponding measurement. Varying the shape of this trapezium as usual does not change the property of point E.

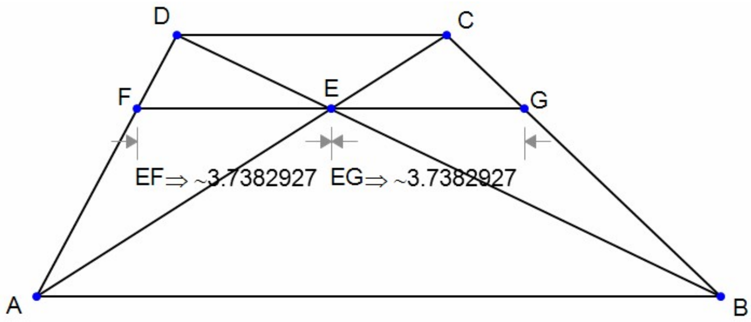

Figure 1. E Midpoint by measurement

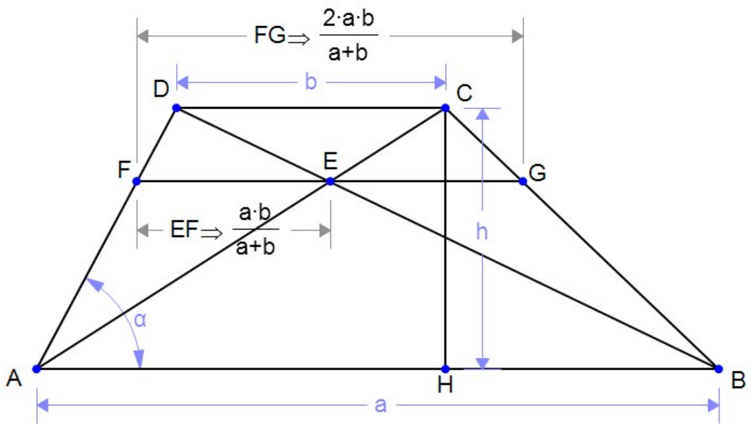

Figure 2. Dependency of midpoint E from length of a and b

But he/she does not know yet how this property depends on the variables, which determine the trapezium's shape. Therefore he/she constrains its basis and its height by variables of length and one of its angle by size (Fig. 2). The result of the automated calculation of the length of FG turns out to be the harmonic mean of the bases and E divides this length and in addition he/she learns that this length is only depending on the lengths of the basis. Now he/she obtains a specific answer for the general "how-question". — After all he/she might try to prove this property of using similarity arguments.

2. EXAMPLES

Now we give some typical applications of the automated calculation using GX (Schumann 2013).

2.1. Example 1: GX as a geometric formula generator

A formula for the area of an octagon constructed by transversals of an arbitrary trapezium is shown in Fig. 3.

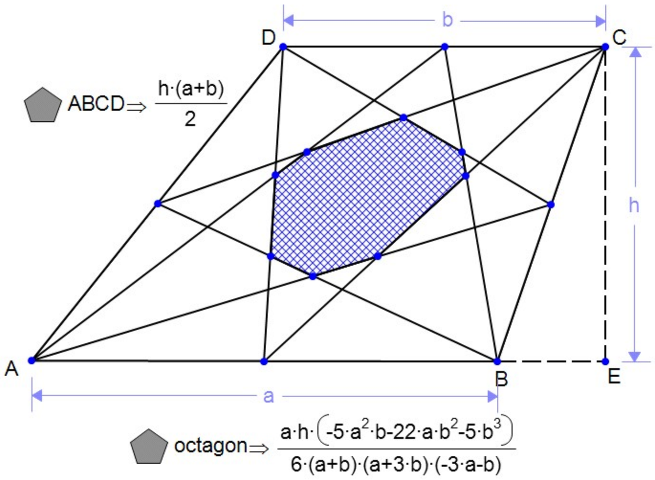

Figure 3. Area formula of a transversal octagon

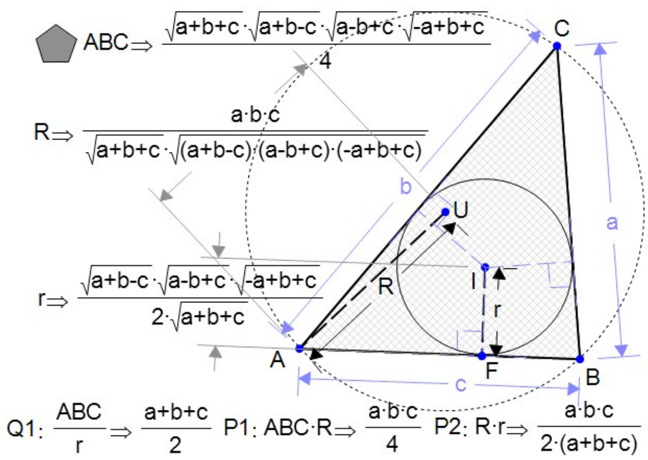

Figure 4. Formula of area, radius of circumcircle and incircle of the triangle

Incircle and circumcircle of a triangle ABC are to be constructed as usual (Fig. 4). New relations can be derived from calculated expressions for r, R and the area by quotients or products.

2.2. Example 2: GX as mean for generalization

By means of the theorem of Routh we can derive rules for the circumference and area of polygons, especially for triangles which are formed by transversals of the triangle ABC. Fig. 5, 6 show the first two diagrams with the "outer" triangles DEF and UVW constructed by the transversals from the vertices A, B and C with respect to the corresponding subdivision points. GX calculates the areas $|ABC|$, $|DEF|$ and $|UVW|$ dependent on a, b and c. By induction the area of these triangles can be found when dividing the triangular sides into $2k$ $(k = 2, 3, \ldots)$ equal parts: $|DEF| = 4|ABC|(k-1)^2/(2k+1)^2, |UVW| = 4|ABC|(k-1)^2/(4k-1)^2$.

An example for the effective combination of the constraining constructional option "tangent" and the algebraic calculation compound of GX is the construction of a chain of circles inside the figure of Arbelos and calculation of the corresponding radii depending on a and b, the radii determining the figure of Arbelos (Fig. 7). By induction we recognize at once the formula

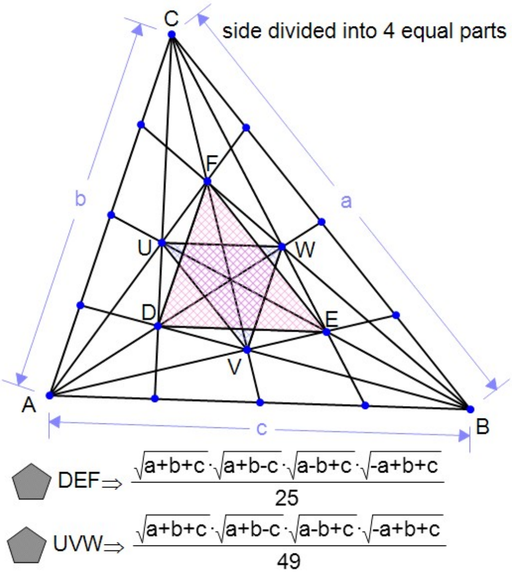

Figure 5. Area of transversal triangle, $k = 2$

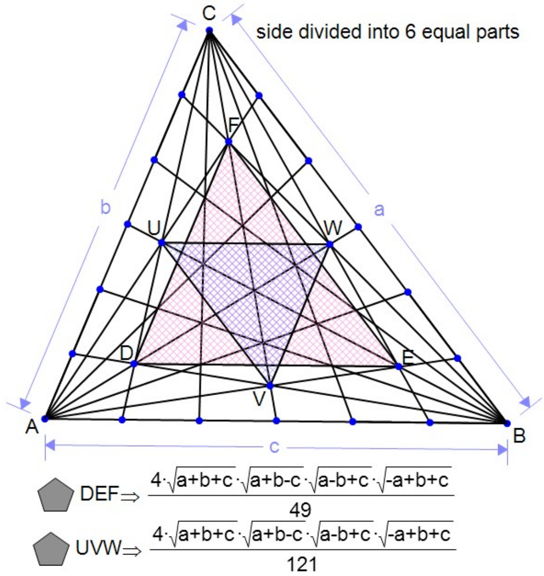

Figure 6. Area of transversal triangle, $k = 3$

for the radii of the chain: $r_n = ab(a+b)/(a^2 + ab + n^2b^2), n = 1, 2, \ldots$

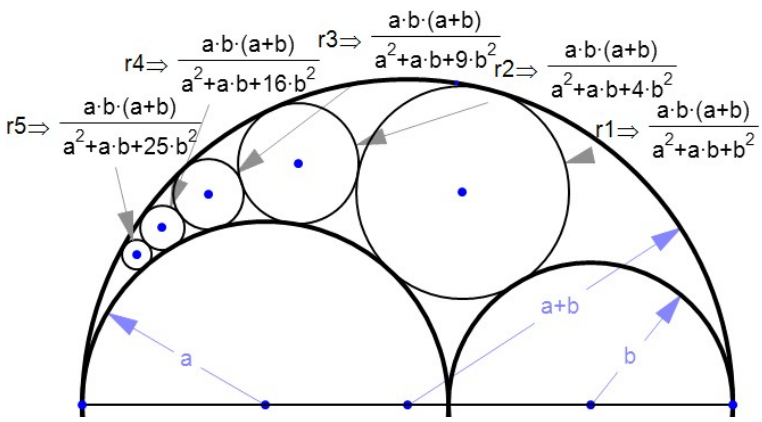

Figure 7. Chain of circles inside Arbelos

2.3. Example 3: GX as mean for new approaches to traditional topics

We have easily constructed the so-called Soddy configuration consisting three mutual tangent circles k_1, k_2 and k_3 and their outside and inside touching circles using the constructional option “tangent” (Fig. 8). In this configuration the determining variables are the radii r_1, r_2 und r_3. The radius r_i and ro of the inside respectively outside touching circle are calculated. By means of transforming these radii replacing the reciprocal radii by the corresponding curvature we obtain the four-circle-theorem of Descartes: The double sum of the squared curvatures of four mutual touching circles equals the squared sum of their curvatures.

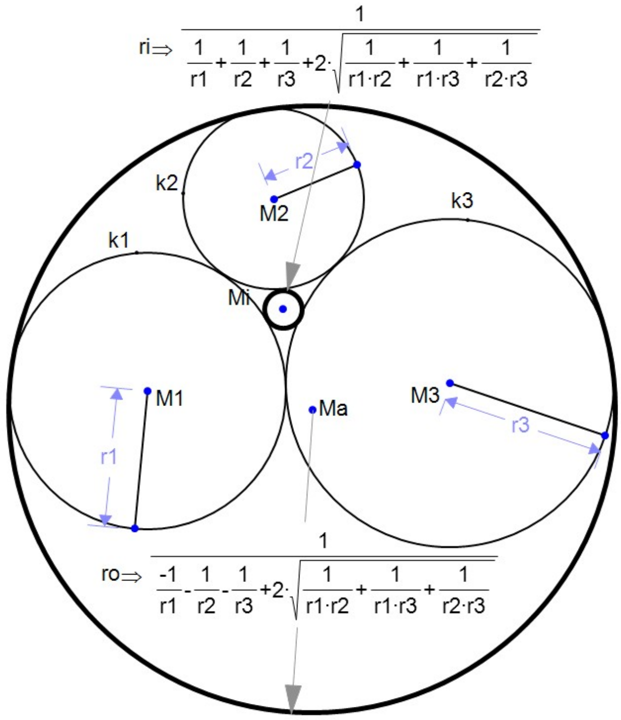

Figure 8. Soddy-configuration with calculated radii of inside and outside touching circles

3. CONCLUSION

The importance of automated algebraic calculations with Geometry Expressions (GX) in combination with constructions beyond ruler and compass consists in the recognition of the algebraic dependencies of variables to be discovered from the determining variables supplementing to the measurements of geometric figures in conventional IDGS. Therefore new possibilities of finding geometric rules are opening up. For this purpose, the system supports the use of heuristic methods such as generalization and analogy observation. But the calculations often lead to relatively complicated terms that must be understood, interpreted and compared (GX offers export opportunities for professional mathematical assistance programs for further processing of expressions). The calculated expressions as formulas represent unproven statements. If not using or applying these statements in an informal way only, there arise corresponding problems of proving, which must be solved within the framework of cognitive psychology called interpolation problems, i. e. a corresponding proof “interpolates” between known prerequisite and known statement. A successful use of the calculation option necessarily requires the answer of the following question: Which variables are sufficient in order to calculate the variables sought? GX assists the user in recognizing overdetermination in the case of underdetermination the system automatically use auxiliary variables.

A major problem when using GX is the violation of the conformity principle of user expectations if the underlying calculation algorithm has no or only a “useless” result. It lies in the nature of things that there does not exist an universal algebraic calculation algorithm, which works for all the constructable geometrical figures. Therefore, it is not advisable to use GX in geometry lessons for open exploration with algebraic calculations. There must a preselection of suitable tasks be made; but this favors a certain selectionism of tasks. Another less serious problem is the unexpected behavior of the drag mode in GX in comparison with that in the conventional IDGS, which however does not impair the algebraic calculation mode. – Nevertheless GX is a valuable new tool to be worth for usage in geometrical education for upper secondary in order to bridge the gap between geometry and algebra. But regarding the curricular importance we cannot match with the content of Philip H. Todd's article in the 71st Yearbook "Understanding Geometry for a Changing World" which seems to be overestimating the significance of GX for learning and teaching plane geometry.

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АВТОМАТИЗИРОВАННЫЕ ВЫЧИСЛЕНИЯ АЛГЕБРАИЧЕСКИХ ВЕЛИЧИН ПРИ ПОСТРОЕНИИ ИНТЕРАКТИВНЫХ ГЕОМЕТРИЧЕСКИХ ФИГУР — ДИДАКТИЧЕСКИЙ АНАЛИЗ

Хайнц Шуман

Аннотация

Обычно интерактивные системы динамической геометрии не могут показать, как измеряемая характеристика геометрической фигуры зависит от других ее параметров. С помощью методов «Automated Deduction in Geometry» (ADG), реализованных в динамической геометрии, становится возможным вычислить какой-нибудь параметр фигуры в зависимости от остальных ее параметров. Поясним это свойство системы с помощью выбранных примеров, которые демонстрируют применение этого инструмента и оценивают возможность его применения в классе.

Ключевые слова: *планиметрия, автоматизированная вычисления, интерактивные динамические системы геометрии.*

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