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APPROXIMATION OF AUTOMATA WITH RESPECT TO THE ANNIHILATION PREDICATE

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Abstract

Automata are the most natural mathematical model for systems that may be in different states. The transfer of questions of approximability with respect to the annihilation predicate of the first kind from varieties of semigroups and polybasic structures to automata is connected with the fundamental property of the automaton — its recognizability. Considering a tribasic semigroup distributive algebra as a model of a semigroup automaton, we have shown that as a result of the transition to polybasic structures, the uniformity of the predicate of equality and the annihilation predicate of the first kind disappears. Consequently, the criteria for decidability of the problem of equality and annihilation predicates will be different. The approximability of automata by tribasic semigroup distributive algebras is associated with the algorithmic decidability of the corresponding problems. The question of the algorithmic decidability of the cancellation problem is of special interest. If there exists an algorithm that determines whether one word is zero for the other for any two words.

Keywords: *automaton, recognizability of automata, predicate of annihilation, semigroup, semigroup algebra, homomorphism of semigroups, algorithmic decidability, approximation of semigroups, finite approximation.*

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1. INTRODUCTION

The divisibility predicate is of especial significance in semigroup theory. In terms of divisibility one defines Green’s relations, simplicity, regularity and its modifications. An important particular case of divisibility relation is the relation of annihilation.

The fundamental order on the set of idempotents in rings and semigroups is defined in terms of the annihilation relation. This relation naturally arises in the study of inverse semigroups, semistructures of groups, and semistructures of nilsemigroups.

A first-type annihilation predicate θ in semigroup S is defined as follows:

$$a, b \in S : (a, b) \in \theta \Longleftrightarrow a \cdot b = b \cdot a = a.$$

Varieties of semigroups finitely approximable with respect to the annihilation predicates are completely described and also the uniformity of the first-type annihilation predicate for the equality predicate in the case of finite approximability of variety of semigroups has been shown [1].

Some questions arise not only about approximability of other complex semigroup structures but also about preserving the uniformity of predicates. The important role the notions of zero and identity of a semigroup play in semigroup theory is well known. These concepts are rather naturally generalized in the notion of a unit-ideal element.

Interest to these elements is defined from the view-point of a unit-ideal subsemigroup (generalization of an ideal) of a semigroup introduced by E. S. Lyapin [2]. They play an important role in the study of dependecies between subsemigroups of a semigroup. In works on approximation of semigroups one often considers theso-called unit-ideal predicates and finds conditions of approximation with relation to them. One of the authors considered these predicates while working with approximation of tribasic semigroup distributive algebras [3, 4].

The annihilation predicate defined this way (sometimes specified first-type) is uniform for the divisibility predicate of unit-ideal elements in approximation of tribasic semigroup structures [5].

2. SEMIGROUP AUTOMATA APPROXIMATION

With the rapid development of Computer Science theoretical constructions in the field of informatics acquired a new look. Some of the well-studied notions such as Turing machine with extensions and restrictions, automata, formal grammars, are not only used in the process of designing and creating software but also help to grasp principle possibilities of computerized solutions for some problems.

Automata are the most natural mathematical model for systems that may be in different states. They not only change under the influence of certain external conditions, but also affect the external environment themselves.

Searching for approximability conditions, as well as finite approximability with respect to various predicates is also relevant in automata theory. Especially, when we consider the connection between approximability and the principle algorithmic decidabilty of the predicate problem. The above-mentioned first-type annihilation predicate, when transferring it from semigroup varieties and polybasic structures to automata, is connected with the fundamental property of the automaton — its recognizability (by Eilenberg [6]).

We will call a four-tuple $(G, F(X), F(Y), g)$ a semigroup automaton, where G is the semigroup of states, $F(X)$ is the semigroup of input words from alphabet X, $F(Y)$ is the semigroup of output words from the alphabet Y, and for the output function $g : G \times F(X) \to F(Y)$ the statements: $g(a, e_X) = e_Y$, $g(a, pq) = g(a, p) \cdot g(a, q)$, $g(ab, p) = g(a, p) \cdot g(b, p)$ hold for all $a, b \in G$, $p \in F(X)$, $q \in F(X)$, where e_X, e_Y — empty words from $F(X)$, $F(Y)$. The given definition of an automaton is narrower in comparison with the classical one, because we consider an undefined automaton state function.

Some properties of particular kinds of automata can be studied with the help of an approximation method that helps to reduce semigroup automata to tribasic semigroup distributive algebras (A, B, C, f), where A, B, C — arbitrary semigroups, and the map $f : A \times B \to C$ has the following properties:

$$f(a_1 \cdot a_2, b) = f(a_1, b) \cdot f(a_2, b), \quad f(a, b_1 \cdot b_2) = f(a, b_1) \cdot f(a, b_2)$$

for all $a, a_1, a_2 \in A, \quad b, b_1, b_2 \in B$.

A triple (α, β, γ) of homomorphisms $\alpha : G \to A, \beta : F(X) \to B, \gamma : F(Y) \to C$ that has the property: $\gamma(g(a, b)) = f(\alpha(a), \beta(b))$, for all $a \in G, b \in F(X)$ is called a homomorphism of automata $(G, F(X), F(Y), g)$ to algebra (A, B, C, f).

An algebras characters class over field P is called a class of algebras of the kind $(M, Hom(M, P^x), P, f_0)$, where M — an arbitrary commutative semigroup and P^x — multiplicative semigroup of the field P, f_0 — natural action of semigroup $Hom(M, P^x)$. Characters over field P are called homomorphisms to algebras of this class.

3. APPROXIMATION OF AUTOMATA WITH RESPECT TO THE ANNIHILATION PREDICATE

A semigroup automaton $(G, F(X), F(Y), g)$ is approximable with respect to a predicate θ in an input language by characters over field P^x, if for any subsets X_1, X_2 of the semigroup $F(X)$ from the domain of θ such that $\theta(X_1, X_2)$ is false, there exists such a character (α, β, γ) over field P^x such that $\theta(\beta(X_1), \beta(X_2))$ is false.

Semigroup automaton $(G, F(X), F(Y), g)$ is approximable with respect to the predicate θ in an output language by characters over field P^x, if for any subsets Y_1, Y_2 of the semigroup $F(Y)$ from the domain of θ such that $\theta(Y_1, Y_2)$ is false, there exists such a character (α, β, γ) over field P^x such that $\theta(\gamma(Y_1), \gamma(Y_2))$ is false.

Theorem. For any field P an arbitrary semigroup automaton $(G, F(X), F(Y), g)$ with a commutative semigroup of states is approximable in an input and output language with respect to the predicate of the annihilation of the first kind θ by characters over field P.

Proof. Let $u, v \in F(X)$ be bound by the annihilation predicate θ and u is not divisible by v.

Then the equalities $u \cdot v = v \cdot u = v$ hold, meaning the elements are unit-ideal.

Introduction of the filter congruence η to $F(X)$ (getting into a convex semigroup) generates a semilattice on the corresponding factor semigroup. Now we consider minimal filters $N(u)$, $N(v)$, containing elements u and v.

Suppose that $v \in N(u)$, then the structure of $N(u)$ implies that there exist q_1, q_2 from $F(X)$ such that $u = q_1 \cdot v \cdot q_2$, but that means that u is divisible by v which is impossible. Therefore $v \notin N(u)$ and then $N(u) \neq N(v)$. Thus, η-classes of elements u and v are different and these elements can be separated by the homomorphism $\beta : F(X) \to \{0, 1\}$, constructed as follows:

$$\forall x \in F(X) \quad \beta(x) = \begin{cases} 0, & \text{if } N(x) \cdot N(u) \neq N(u), \\ 1, & \text{if } N(x) \cdot N(u) = N(u). \end{cases}$$

Here $\beta(u) = 1$, $\beta(v) = 0$, implying that $\beta(u)$ is not divisible by $\beta(v)$.

Obviously, $\{0, 1\}$ is contained in $Hom(G, P^x)$ because the semigroup of states is commutative. Then the semigroup $F(X)$ is approximable with respect to the predicate θ not only in $\{0, 1\}$, that is finitely.

Also and consequently for any field P the semigroup automaton $(G, F(X), F(Y), g)$ is approximable with respect to the predicate θ in the input language by homomorphisms to $(G, Hom(G, P^x), P, f_0)$ by characters over the field P.

Similarly, one can show that the semigroup automaton $(G, F(X), F(Y), g)$ is approximable in the output language with respect to the annihilation predicate of the first kind θ by characters over the field P.

Full description of varieties of semigroups, finitely approximable with respect to the predicates of annihilation [1] shows that in the case of finite approximability of varieties of semigroups the predicate of annihilation of the first kind is uniform for the equality predicate.

Conditions of approximability of an automaton with respect to the equality predicate in input and output alphabets are known [7]. For a semigroup automaton to be approximable, with respect to the equality predicate in an input (output) alphabet by characters, the commutativity and separablility of the semigroup of input (output) automaton words is necessary and sufficient condition.

4. CONCLUSION

Considering a tribasic semigroup distributive algebra as a model of a semigroup automaton, we have shown that as a result of the transition to polybasic structures, the uniformity of the predicate of equality and the predicate of the annihilation of the first kind disappears. Consequently, the criteria for decidability of the problem of equality and annihilation predicates will be different.

Therefore, the question of the algorithmic decidability of the cancellation problem is of a special interest, i.e., if there exists an algorithm that determines whether one word is zero for the other for any two words. Generally, the answer is negative [8]. But the question is still open for complex and polybasic semigroup algebraic structures and, therefore, for automata.

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