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THE SQUARED ABSOLUTE VALUE OF $(a+bi)+\sqrt[2^n]{c+di}$

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Abstract

The squared absolute value of $A=(a+bi+\sqrt{c+di})$ can be expressed simply by $A\bar{A}=(a+bi+\sqrt{c+di})(a-bi+\sqrt{c-di})$, so by roots of complex numbers. Because this expression is real we are interested in an expression, if it exists, that uses only square roots of positive numbers. This expression with only roots of positive numbers is also given for the more general case with $(a+bi+\sqrt[2^n]{c+di})$ with 2^n-th roots.

Keywords: *absolute value, 2^n-th root, complex number.*

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1. INTRODUCTION

In this paper I present a formula for the squared absolute values of complex numbers of the form

$$A=(a+bi)+\sqrt[2^n]{c+di}.$$

Of course this value is just: $A\bar{A}=\left((a+bi)+\sqrt[2^n]{c+di}\right)\left((a-bi)+\sqrt[2^n]{c-di}\right)$.

But for 2^n-th roots this squared absolute value can also expressed by square roots of *positive* numbers, without using roots of complex numbers.

In section 2 and 3 I present these expressions for 2nd and 4th roots, in section 4 for the general case of 2^n-th roots.

In section 5 I give the polynomial equations for the squared absolute values.

Definition: As usual with $\sqrt[k]{x}$ the principal value of the root is denoted, i.e. $-\pi/k<\arg(\sqrt[k]{x})\le\pi/k$.

2. THE SQUARED ABSOLUTE VALUE OF $(a+bi)+\sqrt{c+di}$

$$A=(a+bi)+\sqrt{c+di}. \tag{1}$$

The squared absolute value can be expressed by roots of complex numbers:

$$|A|^2=A\bar{A}=\left(a+bi+\sqrt{c+di}\right)\left(a-bi+\sqrt{c-di}\right), \tag{2}$$

$$|A|^2=(a^2+b^2)+(a-bi)\sqrt{c+di}+(a+bi)\sqrt{c-di}+\sqrt{c^2+d^2}. \tag{3}$$

For the rhs of (3) it is obvious that it is real. Therefore we are interested in an expression that uses only square roots of real, positive numbers.

$$|A|^2 = (a^2+b^2) + \sqrt{c^2+d^2} \pm \sqrt{2(a^2+b^2)\sqrt{c^2+d^2} + 2(a^2-b^2)c + 4abd} \tag{4}$$

take the positive square root, the principal value of $\sqrt{c^2+d^2}$

Example:
$|(1+i)+\sqrt{2+i}|^2 = 2+\sqrt{5}+\sqrt{4\sqrt{5}+4}$.

Now I prove, that the rhs of (4) and the arguments of all square roots and so the square roots itself are real and ≥ 0 without using the fact that the rhs is the squared absolute value of a complex number.

$$B = (a^2+b^2)+\sqrt{c^2+d^2},\ C = (a^2+b^2)\sqrt{c^2+d^2},\ D = 2(a^2-b^2)c+4abd,$$

$$|A|^2 = B + \sqrt{2C+D} \tag{5}$$

$$(2C)^2 = D^2 + 4\big((a^2-b^2)d - 2abc\big)^2 \qquad \to 2C \geq |D|,$$

$$\to 2C + D \geq 0 \text{ the argument of the square root in (5).} \tag{6}$$

$$B^2 = \sqrt{2C+D}^2 + (c-a^2+b^2)^2 + 2(d-2ab)^2 \qquad \to B \geqslant |\sqrt{2C+D}|,$$

$$\to B + \sqrt{2C+D} \geq 0. \tag{7}$$

3. THE SQUARED ABSOLUTE VALUE OF $(a+bi)+\sqrt[4]{c+di}$

$$A = (a+bi)+\sqrt[4]{c+di}, \tag{8}$$

$$|A|^2 = (a^2+b^2)+\sqrt[4]{c^2+d^2}$$
$$\pm\sqrt{2(a^2+b^2)\sqrt[4]{c^2+d^2} \pm \sqrt{2(a^2+b^2)^2\sqrt{c^2+d^2} + 2(a^2-b^2)\big((a^2-b^2)c+4abd\big) - 8a^2b^2c}} \tag{9}$$

take the positive square root of $\sqrt{c^2+d^2}$ and $\sqrt[4]{c^2+d^2}$.

Examples:

$$\left|(1+i)+\sqrt[4]{2+i}\right|^2 = 2+\sqrt[4]{5}+\sqrt{4\sqrt[4]{5}+\sqrt{8\sqrt{5}-16}}$$

$$\left|i+\sqrt[4]{2+i}\right|^2 = 1+\sqrt[4]{5}+\sqrt{2\sqrt[4]{5}-\sqrt{2\sqrt{5}+4}}$$

↑ why the — sign at the 2nd nested root see figure 1

$$\left|-1+\sqrt[4]{2+i}\right|^2 = 1+\sqrt[4]{5}-\sqrt{2\sqrt[4]{5}+\sqrt{2\sqrt{5}+4}}$$

↑ why the — sign at the 1st nested root see figure 1

Similar to section 1 now I prove, that the rhs of (9) and the arguments of all square roots and so the square roots itself are real and ≥ 0 without using the fact that the rhs is the squared absolute value of a complex number.

$$B = (a^2+b^2)+\sqrt[4]{c^2+d^2}, \quad C = (a^2+b^2)\sqrt[4]{c^2+d^2},$$
$$D = 2(a^2-b^2)\big((a^2-b^2)c+4abd\big) - 8a^2b^2c,$$

$$|A|^2 = B + \sqrt{2C+\sqrt{2C^2+D}}. \tag{10}$$

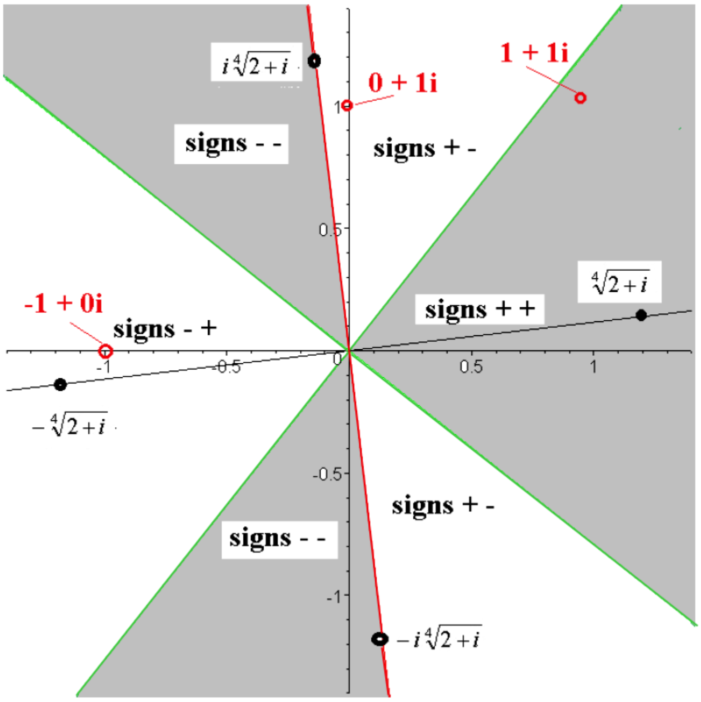

Figure 1. $n=2$, the signs of the nested roots for $(a+bi)+\sqrt[4]{2+i}$ The red line and the 2 green lines separate the 6 regions with different signs of the nested square roots, crossing the red line changes the 1st sign and crossing a green line changes the 2nd sign

$$
\begin{aligned}
&(2C^2)^2 = D^2 + 4\left((a^4-6a^2b^2+b^4)d - 4abc(a^2-b^2)\right)^2 \quad \to 2C \geq |D|,\\
&\to 2C^2 + D \geq 0 \text{ the argument of the innermost square root in (5).}
\end{aligned} \tag{11}
$$

The proof that the argument of the remaining outermost root und the complete value are positive is left for the reader. But this is shown also in section 4 for the general case by construction.

The specialisation $c+di \to (c+di)^2$ i.e. $c \to c^2-d^2,\ d \to 2cd,\ c^2+d^2 \to (c^2+d^2)^2$ transforms the innermost square root $\sqrt{2C^2+D}$ into the D of section 1. So we get as specialisation of (9) the formula (4).

4. THE GENERAL CASE, THE SQUARED ABSOLUTE VALUE OF $(a+bi)+\sqrt[2^n]{c+di}$

$$
B = (a^2+b^2) + \sqrt[2^n]{c^2+d^2}, \quad C = (a^2+b^2)\sqrt[2^n]{c^2+d^2},
$$
$$
D = 2\,\mathrm{Re}\left((a+bi)^{2^n}(c-di)\right) \quad \text{for } n \geq 0.
$$

with n nested square roots

$$
|A|^2 = B \pm \sqrt{2C \pm \sqrt{2C^2 \ldots \pm \sqrt{2C^{2^{n-2}} \pm \sqrt{2C^{2^{n-1}} + D}}}}. \tag{12}
$$

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$

depending on a, b, c, d the signs of these nested roots can be + or −

For the special case $n=0$ i.e. the first root:

$$\begin{aligned} B &= (a^2+b^2)+c^2+d^2, \\ D &= 2\,\mathrm{Re}\big((a+bi)(c-di)\big) = 2(ac+bd), \\ |A|^2 &= (a+bi)+(c+di) = (a+c)+i(b+d) = B+D = (a+c)^2+(b+d)^2. \quad \text{o.k.} \end{aligned} \tag{13}$$

Proof by induction on the integer n:

— for $n=0$ (12) is correct, see the special case in (13),
— assume (12) is correct for n, i.e.

$$\begin{aligned} D(2^n) &= 2\,\mathrm{Re}\big((a+bi)^{2^n}(c-di)\big) = (a+bi)^{2^n}(c-di)+(a-bi)^{2^n}(c+di), \\ D(2^n)^2 &= 2(a^2+b^2)^{2^n}(c^2+d^2)+(a+bi)^{2^{n+1}}(c-di)^2+(a-bi)^{2^{n+1}}(c+di)^2 \end{aligned} \tag{14}$$

now replace $(c+di)^2$ by $(c+di)$, $(c-di)^2$ by $(c-di)$ and

c^2+d^2 by $\sqrt{c^2+d^2}$, this replacement is marked by the index $_R$ (15)

$$\begin{aligned} {}_RD(2^n)^2 &= 2(a^2+b^2)^{2^n}\sqrt{c^2+d^2}+(a+bi)^{2^{n+1}}(c-di)+(a-bi)^{2^{n+1}}(c+di) = \\ &= 2(a^2+b^2)^{2^n}\sqrt{c^2+d^2}+D(2^{n+1}), \\ {}_RD(2^n) &= \sqrt{2(a^2+b^2)^{2^n}\sqrt{c^2+d^2}+D(2^{n+1})}. \end{aligned} \tag{16}$$

Now apply the replacement (15) to $|A^2|$ in (12), the resulting ${}_RD(2^n)$ in the innermost nested square root replaced by (16). This is (12) now for the $(n+1)$-th root. q.e.d.

Analogous to a continued fraction, for which $(a_0,a_1,\ldots,a_{k-1},a_k)_{1/x}$ is defined as:

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\ldots\cfrac{1}{a_{k-1}+\cfrac{1}{a_k}}}}$$

we now define a “continued square root” $(a_0,a_1,\ldots,a_{k-1},a_k)_{\sqrt{x}}$ as:

$$a_0+\sqrt{a_1+\sqrt{a_2+\ldots\sqrt{a_{k-1}+\sqrt{a_k}}}}\,.$$

So the squared absolute value of $(a+bi)+\sqrt[2^n]{c+di}$ in (12) can be written in a very compact manner as continued square root:

$$\begin{aligned} |A|^2 &= (B+D)_{\sqrt{x}} && \text{for } n=0, \\ |A|^2 &= (B,2C+D)_{\sqrt{x}} && \text{for } n=1, \\ |A|^2 &= \Big(B,\underbrace{2C,2C^2,\ldots,2C^{2^{n-2}},2C^{2^{n-1}}+D}_{n}\Big)_{\sqrt{x}} && \text{for } n\geq 2. \end{aligned} \tag{17}$$

For the later use, now I assign the names $R_1,\ldots,R_n$ to the nested square roots:

$$|A|^2 = B \pm \underbrace{\sqrt{2C \pm \underbrace{\sqrt{2C^2 \ldots \pm \underbrace{\sqrt{2C^{2^{n-2}} \pm \underbrace{\sqrt{2C^{2^{n-1}}+D}}_{R_n}}}_{R_{n-1}}}}_{R_2}}}_{R_1} \tag{18}$$

Some examples for $n=3$, i.e. 8th roots:

$$\left|1+\sqrt[8]{2+i}\right|^2 = 1+\sqrt[8]{5}+\sqrt{2\sqrt[8]{5}+\sqrt{2\sqrt[4]{5}+\sqrt{2\sqrt{5}+4}}}$$

$$\left|(-1+i)+\sqrt[8]{2+i}\right|^2 = 2+\sqrt[8]{5}\underset{\uparrow}{-}\sqrt{4\sqrt[8]{5}\underset{\uparrow}{-}\sqrt{8\sqrt[4]{5}\underset{\uparrow}{-}\sqrt{32\sqrt{5}+64}}}$$ see figure 2

$$\left|(-1-i)+\sqrt[8]{2+i}\right|^2 = 2+\sqrt[8]{5}\underset{\uparrow}{-}\sqrt{4\sqrt[8]{5}+\sqrt{8\sqrt[4]{5}\underset{\uparrow}{-}\sqrt{32\sqrt{5}+64}}}$$ see figure 2

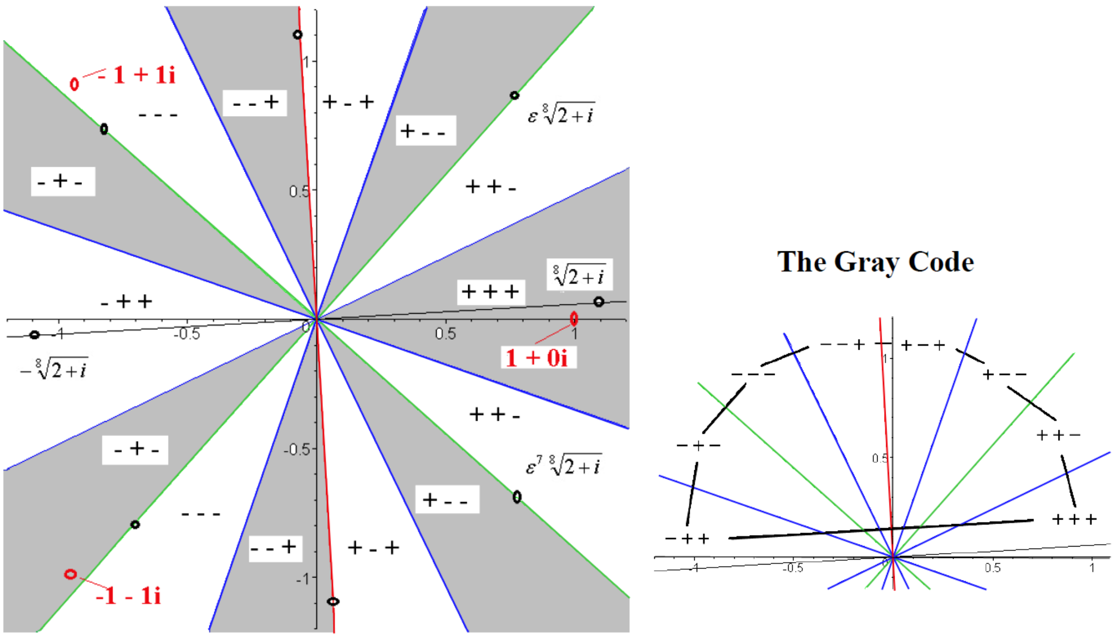

Figure 2. $n=3$ the signs of the nested roots for $(a+bi)+\sqrt[8]{2+i}$ "a" at the horizontal axis, "b" at the vertical axis. The red line, the 2 green lines and the 4 blue lines separate the 14 regions with different signs of the nested square roots, crossing the red line changes the 1st sign, crossing a green line changes the 2nd sign and crossing the blue lines changes the 3rd sign, ε is an 8th root of unit, $\varepsilon = e^{2\pi i}/8 = (1+i)/\sqrt{2}$

The sign sequence of length 8 in the upper (and also in the lower) half plane: $+++$, $++-,\ldots,-+-,-++$ forms a so called Gray code.

Explication and proof for figure 2, just a sketch, I leave the details for the reader.

The nested square roots $R_1,...,R_n$ in (18) can be expressed by:

$$R_k = 2\,\mathrm{Re}\big((a+bi)^{2^{k-1}}\sqrt[2^{n+1-k}]{c-di}\big),$$
$$R_n = \pm\sqrt{2C^{2^{n-1}}+D} \quad R_k = \pm\sqrt{2C^{2^{k-1}}+R_{k+1}} \text{ for } 1 \leqslant k < n. \tag{19}$$

Crossing one of the lines in figure 2 changes the sign of one of the nested square roots R_k. So on each of the red, green, blue lines a $R_k = 0$.

$$R_k = 0 \to (a+bi)^{2^{k-1}}\sqrt[2^{n+1-k}]{c-di} = r_1 i, \qquad r_1, r_2 \text{ are real}$$
$$\to (a+ib) = \sqrt[2^n]{c+di}\sqrt[2^{k-1}]{i r_2},$$
$$\to \arg(a+bi) = \arg(\sqrt[2^n]{c+di}) + \pi/2^k m \qquad \text{odd } m = 1,\dots,2^k-1 \ (2) \tag{20}$$

5. THE POLYNOMIAL EQUATION FOR $\left|(\boldsymbol{a}+\boldsymbol{bi})+\sqrt[2^n]{\boldsymbol{c}+\boldsymbol{di}}\right|^2$

Similar to section 4 where a continued square root was defined here a "continued square" $(a_0, a_1, \dots, a_{k-1}, a_k)_{x^2}$ is defined as:

$$a_0 + \left(a_1 + \left(a_2 + \dots (a_{k-1} + (a_k)^2)^2\right)^2\right)^2.$$

Just by moving in (12) B from the rhs as $-B$ to the lhs, squaring both sides, moving $-2C$ to the lhs ..., we get the polynomial equation of degree 2^n for $x = |A|^2$ expressed as continued square:

$$\Big(-D, \underbrace{-2C^{2^{n-1}}, -2C^{2^{n-2}}, \dots -2C^2, -2C}_{n-1}, x-B\Big)_{x^2} \tag{21}$$

The polynomial equation of degree 2^{n-1} for $y = (|A|^2 - B)^2$ expressed as continued square is:

$$\Big(-D, \underbrace{-2C^{2^{n-1}}, -2C^{2^{n-2}}, \dots -2C^2, -2C+y}_{n-1}\Big)_{x^2} \tag{22}$$

Example for $n = 3$, i.e. 8th roots, the equation of degree 4 for y is:

$$y^4 - 8Cy^3 + 20C^2y^2 - 16C^3y + 4C^4 - D$$

with C, D for $n = 3$.

For $n = 4$, i.e. 16th roots, the equation of degree 8 for y is:

$$y^8 - 16Cy^7 + 104C^2y^6 - 352C^3y^5 + 660C^4y^4 - 672C^5y^3 + 336C^6y^2 - 64C^7y + 4C^8 - D$$

with C, D for $n = 4$.

Without proof:
The Galois groups for general C, D of these equations of degree 2^{n-1} in $\boldsymbol{y}$ are $G(n) = C_{2^{n-1}} : \mathrm{Aut}(C_{2^{n-1}})$.

Here C is the cyclic group with the given index, ":" means group extension, Aut( ) is the automorphisms group, the order of $G(n)$ is 2^{2n-3}.

The Galois groups of equations of degree 2^n in $\boldsymbol{x}$ are $C_2 \times G(n) = C_2 \times C_{2^{n-1}} : \mathrm{Aut}(C_{2^{n-1}})$.

This is also the Galois group of the algebraic number A.

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