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ELLIPTIC INTEGRALS, FUNCTIONS, CURVES AND POLYNOMIALS

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Abstract

The extensive subject of elliptic integrals, functions and curves, being at the junction of analysis, algebra and geometry, has numerous applications in mechanics and physics. Two approaches to the study of elliptic functions have become classical, namely that of Jacobi and that of Weierstrass. Two separate chapters were devoted to these two approaches in the (well-known) course of modern analysis by Whittaker and Watson, without attempting to unite them [1, §§XX, XXII]. Also, two separate chapters are devoted to these two approaches in the latest version (1.0.22 on March 15, 2019) of the NIST Digital Library of Mathematical Functions [2, §§22, 23]. An wide-spread inculcation "explained" that the Weierstrass approach is more suitable for theoretical research, whereas the Jacobi elliptic functions are more common in applications. But, in fact, this dichotomy is artificial, and studying elliptic functions and curves may (and must) be combined in an algebraic approach, establishing a canonical "essential" elliptic function which linear fractional "symmetry" transformations acquire the simplest forms. Although such a natural and fundamental object to be (rightly) called *the Galois essential elliptic function*, was introduced only recently (already in our millennium), its use has quickly become fruitful, not only and not so much for the effective recovery of known results but also for achieving new calculations that once seemed too cumbersome to pursue. The methodological significance of this natural algebraic approach, which undoubtedly transcends back to the (revolutionary) contribution of Galois, is clearly manifested by its application to several fundamental problems of classical mechanics with the achievement of non-standard, capacious and highly efficient solutions.

Keywords: *Galois elliptic function, linear fractional transformation, elliptic and coelliptic polynomial, modular polynomial symmetry.*

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1. THE GALOIS ELLIPTIC FUNCTION

Definition 1. *For a given parameter $\alpha \in \mathbb{C} \setminus \{-2/3, 2/3\}$, the Galois essential elliptic function $\mathscr{R}_\alpha$ is defined as the solution of the differential equation*

$$y'^2 = 4\,y\left(y^2 + 3\,\alpha\,y + 1\right), \tag{1}$$

with a (double) pole at the origin [3–6].

The essential elliptic function $\mathscr{R}_\alpha$ differs from the Weierstrass elliptic function $\wp_\alpha$ by an additive constant. Evidently, if $\wp_\alpha$ is the Weierstrass elliptic function that satisfies the differential equation

$$\begin{aligned} y'^2 &= 4\left(y^3 - \left(3\,\alpha^2 - 1\right)\,y - \alpha\left(1 - 2\,\alpha^2\right)\right) = \\ &= 4(y - \alpha)\left(y - (\alpha - \beta)\right)\left(y - (\alpha - 1/\beta)\right), \end{aligned} \tag{2}$$

$$\beta := (3\,\alpha + d)/2, \quad d^2 = 9\,\alpha^2 - 4$$

then

$$\mathscr{R}_\alpha = \wp_\alpha - \alpha$$

. In particular, when $\alpha = 0$, the essential elliptic function $\mathscr{R}_0$ coincides with the Weierstrass function $\wp_0$. One ought to note that the discriminant of either the cubic polynomial, on the right-hand side of equation (1), or the cubic polynomial on the right-hand side of equation (2), being the product of the squares of the three pairwise differences of three roots, is $d^2 = \left(\beta - 1/\beta\right)^2$, so it does not vanish since, by assumption, $\alpha \neq \pm 2/3$.

We may extend the fundamental domain by adding two functions, corresponding to the two excluded values $\alpha = \pm 2/3$, for which equation (1) degenerates into the equation

$$y'^2 = 4\,y\left(y \pm 1\right)^2,$$

and we, thus, regards the functions ctg^2 and cth^2 as functions corresponding to $\alpha = 2/3$ and $\alpha = -2/3$, respectively, where

$$\mathrm{ctg}^2(x) := -\left(\frac{e^{2\sqrt{-1}x} + 1}{e^{2\sqrt{-1}x} - 1}\right)^2, \quad \mathrm{cth}^2(x) := \left(\frac{e^{2x} + 1}{e^{2x} - 1}\right)^2.$$

Definition 2. *The Galois alternative elliptic function $\mathscr{S}_\beta$ is the vanishing at zero solution of the differential equation*

$$y'^2 = \left(1 - \beta\,y^2\right)\left(1 - y^2/\beta\right),$$

which leading Maclaurin series coefficient is 1 [3–6].

Designating the Jacobi elliptic sine function with $\mathrm{sn}_\beta(\cdot) = \mathrm{sn}(\cdot, \beta)$, where β is the elliptic modulus, we have

$$\mathscr{S}_\beta(x) = \sqrt{\beta}\,\mathrm{sn}_\beta\left(x/\sqrt{\beta}\right),$$

and the square of an alternative elliptic function coincides with the inverse of an essential elliptic function. Precisely, we have

$$\mathscr{S}_\beta^2 = \mathscr{R}_{-\alpha}^{-1}. \tag{3}$$

The alternative elliptic function $\mathscr{S}_\beta$, as the Jacobi elliptic sine sn_β, is an odd function. However, in contrast to the Jacobi elliptic sine, the alternative elliptic function is invariant under the inversion of its elliptic modulus $\beta \mapsto 1/\beta$, while is unable to withstand the sign flip $\beta \mapsto -\beta$, which subjects it to an "imaginary" homothetic transformation, that is,

$$\mathscr{S}_{-\beta}(x) = \sqrt{-1}\,\mathscr{S}_{\beta}\left(x/\sqrt{-1}\right).$$

The pair of degenerate elliptic functions $\mathscr{S}_{\pm 1}$ corresponds to the pair $\alpha = \pm 2/3$, respectively

$$\mathscr{S}_{-1}(x) = \tan(x) := \frac{e^{2\sqrt{-1}x} - 1}{\sqrt{-1}\left(e^{2\sqrt{-1}x} + 1\right)}, \quad \mathscr{S}_{1}(x) = \mathrm{tgh}(x) := \frac{e^{2x} - 1}{e^{2x} + 1}.$$

2. A FUNCTIONAL RATIO

Consider two functional ratios, containing three Jacobi elliptic functions, namely,

$$T_{\pm}(x,\beta) := T\left(x, \frac{1-\beta}{1+\beta}, \sqrt{-\beta} \pm \sqrt{-1/\beta}\right),$$

$$T(x,\beta,\gamma) := \frac{2\,\mathrm{sn}(\gamma x,\beta)/\gamma}{\mathrm{cn}(\gamma x,\beta) + \mathrm{dn}(\gamma x,\beta)}.$$

For a fixed elliptical module β, one of the two ratios $T_{\pm}(x,\beta)$ coincides identically (that is, for all values of its first argument x) with the Galois alternative elliptic function $\mathscr{S}_{\beta}(x)$. Of course, such a coincidence remains true in the degenerate cases, so, in particular, β tending to zero

$$T_{\pm}(x,\beta \approx 0) \approx \frac{\sqrt{-\beta}\,\mathrm{tgh}(x/\sqrt{-\beta})}{\mathrm{sech}(x/\sqrt{-\beta})} = \sqrt{\beta}\sin\left(\frac{x}{\sqrt{\beta}}\right),$$

reflects the fact that the vanishing of the elliptic module β corresponds to the Jacobi elliptic sine sn_0, which coincides with the trigonometric function sin.

We also have

$$T_{+}(x,1) = \frac{\sin(\sqrt{-1}\,x)}{\sqrt{-1}\cos(\sqrt{-1}\,x)} = \mathrm{tgh}(x) = \mathscr{S}_{1}(x),$$

$$T_{-}(x,-1) = \sqrt{-1}\,T_{+}\left(x/\sqrt{-1}, 1\right) = \tan(x) = \mathscr{S}_{-1}(x).$$

The said ratios, which disguise the Galois alternative elliptic function, arise upon calculating integrals of elliptic functions, as discussed earlier this year at the PCA 2019 conference [3].

3. MODULAR POLYNOMIAL SYMMETRIES

The calculation of the roots of the moduar equation of level p is tightly intertwined with calculating the p-torsion points of a corresponding elliptic curve, and the establishment of such remarkable relationship must be attributed entirely and solely to Galois [5, 7–9]. Following Galois, we arrive at a new class of infinite modular polynomial symmetries, corresponding to odd primes, the first of which corresponds to the odd prime 3, and we state it explicitly here.

Problem 1. *Let γ_4 be a given root of*

$$p_4(x) := x^4 + 4\,\alpha\,x^3 + 2\,x^2 - 1/3,\ \alpha \in \mathbb{C} \setminus \{\pm 2/3\},$$

and put

$$p_3(x) := x^3 + \left(1/\gamma_4^2 - 4\right)x + 2\gamma_4.$$

Then, for any root ξ of the polynomial p_3 and for any root $\gamma \neq \gamma_4$ of the polynomial p_4, that is to say $p_3(\xi) = p_4(\gamma) = 0$, the equality

$$\xi^9 \left(\frac{p_4(1/\xi)}{p_4(\xi)}\right)^2 = -2\gamma_4 \left(\frac{\gamma^3\,p_3(1/\gamma)}{p_3(\gamma)}\right)^2$$

holds.

Thus, three values on the left-hand side, corresponding to three roots ξ, and three values of the right-hand side, corresponding to three roots γ, coincide with one and the same (invariant) value which, in fact, lies in the field of rational functions of γ_4.

Amusingly, preliminary attempt for providing a direct "computer" proof to this (correct) identity, at the 17$^{\text{th}}$ International Workshop on Computer Algebra [10], as presented in a talk given by S. Meshveliani (PSI RAS), turned out being erroneous (according to the author of the proof). Soon thereafter, he sent me a revised statement of his algorithm, based on a skillful implementaation of Gröbner bases techniques, yet without a specific implementable code (and no subsequent publication ensued). An elementary proof (without resorting to a computer) was found by an independent researcher Helmut Ruhland [11], with whose permission it was presented at a joint CMC MSU and CC RAS seminar [8]. The case, corresponding to the second odd primes 5, is also worthwhile to explicitly present here, as it resonates with solving the quintic, as discussed at the past PCA 2018 conference [7].

Problem 2. *Let γ_6 be a given root of*

$$p_6(x) := x^{12} + \frac{62\,x^{10}}{5} - 21\,x^8 - 60\,x^6 - 25\,x^4 - 10\,x^2 + \frac{1}{5} +$$
$$+\,\alpha\,x^3\left(x^8 + 4\,x^6 - 18\,x^4 - \frac{92\,x^2}{5} - 7\right) + \alpha^2 x^4\left(\frac{x^6}{5} - 3\,x^2 - 2\right) - \frac{\alpha^3 x^5}{5},$$
$$\alpha \in \mathbb{C} \setminus \{\pm 8\},$$

and put

$$p_5(x) := x^5 + \left(4 + 3\lambda^2 - 10\mu + \lambda\alpha\right)x^3 - 2\left(\lambda + 2\lambda\mu + 2\mu\alpha\right)x^2 +$$
$$+\left(2\lambda^2 - 12\mu + 5\mu^2 + \lambda\mu\alpha\right)x + 2\lambda\mu,$$

where

$$\lambda := \gamma_6 + \bar{\gamma}_6,\ \mu := \gamma_6\bar{\gamma}_6,\ \bar{\gamma} := \frac{(\gamma^2 - 1)^2}{\gamma(4\gamma^2 + \alpha\gamma + 4)}.$$

Then, for any root ξ of the polynomial p_5 and for any root $\gamma \neq \gamma_6 \neq \bar{\gamma}$ of the polynomial p_6, that is to say $p_5(\xi) = p_6(\gamma) = 0$, the equality

$$\xi^{25}\left(\frac{p_6(1/\xi)}{p_6(\xi)}\right)^2 = -2\,\lambda\,\mu\left(\frac{\gamma^5\,p_5(1/\gamma)}{p_5(\gamma)}\right)^2\left(\frac{\bar{\gamma}^5\,p_5(1/\bar{\gamma})}{p_5(\bar{\gamma})}\right)^2$$

holds.

Proving modular polynomial symmetries, along with carrying out highly-efficient arithmetic on elliptic curves, is based on the concepts of elliptic and coelliptic polynomials, which were first announced at the PCA 2014 conference [9], and subsequently rediscussed [5, 7, 8].

4. APPLICATIONS TO CLASSICAL MECHANICS

Over a century ago, we were (most rightfully) told by Alfred George Greenhill "that we may take the elliptic functions as defined by pendulum motion" [12]. Yet not before the turn of the 21^{st} century did we realize that the motion of the simple pendulum (whether oscillatory or rotary) is indeed entirely and solely representable by the Galois essential elliptic function. It (faithfully) maps the (physical) time to the "configuration space" of the pendulum, which might be representable as a (unit) circle in the complex plane. We merely needed to understand that the essential elliptic function maps the (physical) time "directly" to a (well-defined) position, which might be expressed as a (single-valued) "exponent" $\exp(\sqrt{-1}\theta)$, with θ denoting the (multivalued) "angle" [6].[1] Francois Lamarche (McGill University) has contributed to "popularize" and spread the essential elliptic function as the general solution to the pendulum motion problem. The simple pendulum "reappears" in the (so-called) critical motion of "Dzhanibekov's flipping wingnut" [13]. Connecting these two most fundamental problems of classical mechanics required an identification of an axis of "generalized" symmetry of a rigid body, which coincides with no axis of inertia whenever the moments of inertia are pairwise distinct. Such as axis, which turned out being orthogonal to the circular sections of the (so-called) MacCullagh ellipsoid of inertia, was justifiably named the *the Galois axis* [14]. Its conception was superseded by a formula for the "generalized" precession, of a freely moving rigid body, as an elliptic integral of the third kind, which integrand is a function not only rational in the (three) moments of inertia but is symmetric,[2] as well. The symbolic expression of such function was guided by Galois [15]. No less naturally, Galois elliptic functions arise in classifying thread equilibria in a linear parallel force field, in both attracting and repelling cases [4, 16]. In fact, employing these functions in calculating of the length of a thread in a linear parallel repelling force field has led to defining the modified arithmetic-geometric mean (MAGM), which calculation need not necessarily be "mediated" via the arithmetic-geometric mean (AGM) but might be carried out "directly", as discussed at the "Mathematics Stack Exchange" [17].

5. CONCLUSION

We are led to predict that the Galois elliptic functions would gradually (yet inevitably) replace all other elliptic functions, commonly associated with a vast range of "naturally" occurring phenomena, not necessarily confined to classical mechanics. The Jacobi and the Weierstrass elliptic functions would then acquire a permanent place in mathematics history textbooks.

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[1] Note that the multivaluedness of the angle θ does not alter the single-valuedness of $\exp(\sqrt{-1}\theta)$.

[2] Realizing the fact that the rate of the "generalized" precession was symmetric in the moments of inertia was, in fact, most crucial in arriving at an explicit expression. The critical motion was then seen to correspond to a constant rate. This rate is now known as the rate with which the Galois axis (uniformly) rotate, whether the "wingnut flips" or not [13].

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