THE CALCULATION OF THE PROBABILITY DENSITY IN PHASE SPACE OF A CHAOTIC SYSTEM ON THE EXAMPLE OF ROTATOR IN THE HARMONIC FIELD

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Abstract

The method of calculating the probability density distribution for chaotic systems described by the equations of classical nonlinear dynamics is proposed. Specific calculations are performed for the rotator in an external harmonic field. The results of calculation of probability density distribution in the cross section of the phase space are compared with the Poincare cross section for the chaotic attractor obtained by numerical solution of the dynamic equations. It is shown that the corresponding quantum problem for a rotator in an external harmonic field in the semiclassical limit leads to equations describing the distribution of probability density in the classical case.

Keywords: *nonlinear dynamics, chaos, chaotic attractor, the density of the probability distribution of States, semiclassical limit.*

1. INTRODUCTION

A characteristic feature of the system described by nonlinear equations is the occurrence of chaos at certain parameters of the system. As noted by G. M. Zaslavsky and R. Z. Sagdeev in one of the first monographs on this topic [1]: "Since the trajectories of particles in phase space become in this case extremely complex and confusing, it is useless to monitor each trajectory separately. Instead, we should consider a set of trajectories that at any time occupy a finite volume of phase space, and the distribution of particles in it is characterized by some density." Calculation the probability density is a standard problem in statistical physics and quantum mechanics. It is interest to formulate a method for calculating the probability density distribution for the simplest systems in which the state described by a strange (chaotic) attractor is realized. As is known in this case, in the chaotic picture of the phase space appear quite certain features most clearly manifested in the Poincare cross section. It is interesting to compare the results of calculating the probability density of such a system with the Poincare cross section, which are obtained by numerically solving differential equations.

In this paper, without trying to view general case, we consider one of the simplest systems for which the state described by a strange attractor is realized — a rotator under the influence of

a harmonic field. We will obtain an equation that determines the probability density distribution of such a system and give the results of numerical calculation. Since it is relatively easy to perform quantum mechanical calculation for this system, we will show that in the quasi-classical limit the equation for the density matrix of the quantum mechanical problem passes into the equation for the probability density distribution in the classical dynamics problem. This limit transition allows us to assert that, if we use the probabilistic approach, the chaos in the problems of classical nonlinear dynamics is predictable to the same extent as the evolution of the system described by the equations of quantum theory is predictable. The considered problem is remarkable because it allows to successfully use it for training purposes. On the one hand, it is simply formulated for a very real physical system and with the use of modern computing environments simply solved numerically. On the other hand, the study of solutions allows us to demonstrate such features of nonlinear dynamics problems as the transition from regular motion to chaotic with adiabatic change of control parameter. Such transitions occur at different values of the control parameter at its increase or decrease, which corresponds to hysteresis phenomena in nonlinear systems [2, 3].

2. PROBLEM STATEMENT. EQUATION OF MOTION

Consider an electric dipole that can make a one-dimensional rotation (Fig. 1). The real such system is two differently charged balls of the same mass, connected by a nonconducting rod fixed on a hinge.
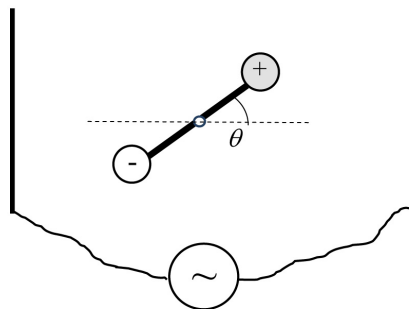

Figure 1

The effect of an external harmonic field can be taken into account by placing such a system between the plates of the capacitor to which the alternating voltage is applied.

We also assume that the rotation is decelerated with the viscous friction force, which is proportional to the angular velocity of rotation. Then the dynamic equation describing the motion has the form:

$$I\ddot{\theta} = qE_0 \sin\theta \cos(\omega t) - \lambda\dot{\theta},$$

where I is the moment of inertia of the dipole, q is the absolute value of charge of the balls, E_0 and ω is the intensity and frequency of the external field, λ is the coefficient of proportionality between the moment of friction and the angular velocity. Here and further the point denotes the derivative of the variable t. To simplify the equation, we perform a large-scale transformation of the time variable $t' = \omega t$, then the equation of motion takes the form:

$$\ddot{\theta} + \gamma\dot{\theta} = f\sin\theta\cos t, \tag{1}$$

where $\gamma = \dfrac{\lambda}{I\omega}, f = \dfrac{qE_0}{I\omega^2}$ (for simplicity, next, use the symbol t instead of t'). Note that this equation and the corresponding Poincare cross section pattern were considered in the monograph [1], and the formulation of similar quantum mechanical problem in the absence of dissipation is considered in [4], where, in particular, it was noted that "systems with harmonic dependence on time are not too popular among theorists". Note that systems with dissipation also "not too popular" in the study of quantum chaos.

The differential equation of the second order (1) is reduced to an autonomous system of three differential equations of the 1st order:

$$\begin{cases} \dot{\theta} = p, \\ \dot{p} = f \sin\theta \cos\tau - \gamma p, \\ \dot{\tau} = 1. \end{cases} \quad (2)$$

The variables θ, p and τ form a three-dimensional phase space. This is the minimum value of dimension for which the trajectory can tend to a chaotic attractor, similar to the Lorentz attractor (see, for example, [5]). At the same time, depending on the control parameter f, the solution can be both regular and chaotic. Parameter γ characterizing the dissipation, usually relies small. However, namely the difference it from zero determines the tendency of the trajectory in phase space to the attractor.

Unlike the time variable, the variable τ can be considered as periodic with a period of 2π. Taking into account the periodicity of the variable θ, it is convenient to consider the trajectory in the phase space as a line "wound" on the torus (Fig. 2).
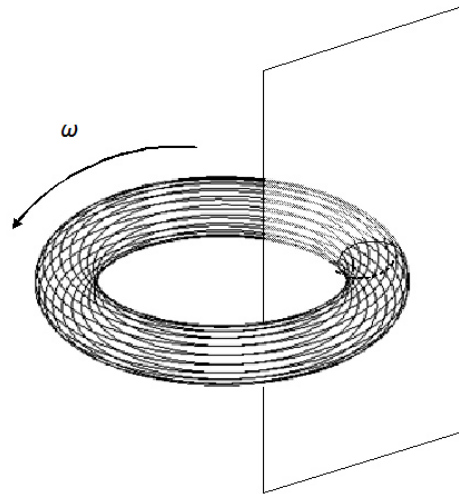

Figure 2

The cross section of such a torus by a plane is a Poincare cross section, the corresponding set of points is a fractal. Numerical calculations for the parameters $f = 3$ and $\gamma = 0.1$ [6] give the fractal dimension approximately equal to 1.7.

2.1. Probabilistic approach

Let us now define a density probability distribution $\rho(\tau, \theta, p)$ as follows: for a given value τ the value $\Delta w = \rho(\tau, \theta, p)\Delta\theta\Delta p$ is equal to the probability that the trajectory of the system passes

in the region $\left[\theta, \theta + \Delta\theta;\ p, p + \Delta p\right]$ and the normalization condition is set:

$$\int\limits_{-\pi}^{\pi} d\theta \int\limits_{-\infty}^{\infty} d p\ \rho(\tau, \theta, p) = 1, \quad \forall \tau. \tag{3}$$

Note that the density of the probability distribution can be approximately obtained by a numerical experiment. For this purpose, the section plane needs to be broken into cells $\Delta\theta\Delta p$, perform calculations $\theta(t)$ and $p(t)$ over large enough time and put ρ for each cell is proportional to the number of points that fell into the cell. It should be noted that in fact the range of values of the variable p is limited: $p \in \left[-p_{\max}, p_{\max}\right]$. Indeed, if in the second equation of the system (2) the effect of the field is replaced by the maximum value f, we obtain the equation: $\dot{p} = f - \gamma p$. The solution of this equation is in explicit form:

$$p(t) = p_0 \exp(-\gamma(t - t_0)) + \frac{f}{\gamma}\left(1 - \exp\left(-\gamma(t - t_0)\right)\right),$$

where $p_0 = p(t_0)$. It is easy to show that $p(t)$ increases only when $p_0 < f/\gamma$. From here we can conclude that

$$|p| \leqslant p_{\max} = f/\gamma. \tag{4}$$

It is interesting to obtain an equation for the probability density distribution explicitly. Note that for Hamiltonian systems, that is, in this case at $\gamma = 0$, this equation has the form (see, for example, [7]):

$$\frac{d\rho}{dt} = [\rho, H] + \frac{\partial\rho}{\partial t}. \tag{5}$$

Here are square brackets are Poisson brackets, in this case:

$$[\rho, H] = \frac{\partial\rho}{\partial\theta}\frac{\partial H}{\partial p} - \frac{\partial\rho}{\partial p}\frac{\partial H}{\partial\theta},$$

and Hamiltonian H has the form:

$$H = \frac{p^2}{2} + f\cos t\cos\theta.$$

To generalize equation (5) to the case of a dissipative system, note that this equation is derived from the equation for an arbitrary function $u(t, \theta, p)$:

$$\frac{du}{dt} = \frac{\partial u}{\partial\theta}\dot{\theta} + \frac{\partial u}{\partial p}\dot{p} + \frac{\partial u}{\partial t} \tag{6}$$

and Hamilton's equations:

$$\dot{\theta} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial\theta}. \tag{7}$$

Adding the dissipation described by the second term on the left side of equation (1), the second of Hamilton's equations (7) can be reduced to the form:

$$\dot{p} = -\frac{\partial H}{\partial\theta} - \gamma p.$$

Substituting this expression into equation (6), we obtain an equation for an arbitrary function $u(t, \theta, p)$. As a result, the equation for the function $\rho(t, \theta, p)$ takes the form:

$$\frac{d\rho}{dt} = p\frac{\partial\rho}{\partial\theta} + f\sin\theta\cos t\frac{\partial\rho}{\partial p} - \gamma p\frac{\partial\rho}{\partial p} + \frac{\partial\rho}{\partial t}.$$

We are interested in a solution approaching a chaotic attractor, which corresponds to the zero value of the total time derivative. The partial time derivative in this case is equivalent to the derivative of the variable τ. Thus, the equation for the function $\rho(t, \theta, p)$ has the form:

$$\frac{d\rho}{d\tau} = p\frac{\partial\rho}{\partial\theta} + f\sin\theta\cos t\frac{\partial\rho}{\partial p} - \gamma p\frac{\partial\rho}{\partial p} = 0. \tag{8}$$

Equation (8) is a linear first-order partial differential equation. From the theory of differential equations (see [8]) it follows that with a similar differential equation, which in General can be written as:

$$\sum_{v=1}^{n} f_v(x_1, x_2, ..., x_n)\frac{d\psi(x_1, x_2, ..., x_n)}{dx_v} = 0, \tag{9}$$

are associated systems of ordinary differential equations. These equations have the form:

$$\dot{x}_v(t) = f_v(x_1, x_2, ..., x_n), \ \ v = 1, 2, ..., n. \tag{10}$$

Characteristic curves defined by the system of equations (10) are associated with solutions of equation (9). In particular, if $\varphi_v(t)$ is the solution of a system of equations (10), then the function $\psi(x_1, x_2, ..., x_n)$ if and only if is the integral of equation (9) when $\psi(\varphi_1(t), \varphi_2(t), ..., \varphi_n(t)) = \text{const}$ for any characteristic curve. It is easy to verify that in the case under consideration the system of equations that determines the characteristic curves of the equation (8) exactly coincides with the system of equations (2) that determines the dynamics of the system of interest.

2.2. Fourier series expansion and numerical computation

One of the methods for solving equations of the form (9) is associated with finding characteristic curves (see [8] for details). In our case it can be implemented as finding functions $\rho(t, \theta, p)$ in the above numerical experiment. However, this is not the only solution.

First of all, we note that for the unique solution of equation (8) the function $\rho(t, \theta, p)$ must satisfy some boundary conditions. In our case, according to inequalities (4), such a boundary condition is:

$$\rho(t, \theta, p_{max}) = \rho(t, \theta, -p_{max}) = 0, \ \ \forall\, \theta, \tau,$$

The fulfillment of this condition, the normalization conditions, which can be rewritten as:

$$\int_{-\pi}^{\pi} d\theta \int_{-p_{max}}^{p_{max}} dp\, \rho(\tau, \theta, p) = 1, \ \ \forall\tau, \tag{11}$$

and the requirements of the periodicity on variable θ and τ uniquely identifies the solution.

Since variables θ, p and τ limited, for the solution of the equation (8) it is convenient to use the decomposition in Fourier series. To make the expansions identical, that is, all variables would be bounded by an interval $[-\pi, \pi]$, let's perform a scale transformation of the variable p:

$$p = p'\frac{p_{max}}{\pi}.$$

As a result, the equation, the boundary condition and the normalization condition take the form (we omitted the prime on the variable p):

$$\frac{d\rho}{d\tau} + \frac{p_{\max}}{\pi} p \frac{\partial\rho}{\partial\theta} + f \frac{\pi}{p_{max}} \sin\theta \cos t \frac{\partial\rho}{\partial p} - \gamma p \frac{\partial\rho}{\partial p} = 0,$$

$$\rho(\tau,\theta,\pi) = \rho(\tau,\theta,-\pi) = 0, \quad \forall\, \theta,\tau,$$

$$\int\limits_{-\pi}^{\pi} d\theta \int\limits_{-\pi}^{\pi} dp\, \rho(\tau,\theta,p) = \pi/p_{max}, \quad \forall\tau.$$

Decomposition of the function $\rho(\tau,\theta,p)$ in a Fourier series written in the form:

$$\rho(\tau,\theta,p) = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \rho_{k,m,q} \exp\left(i(k\tau + m\theta + qp)\right).$$

The differential equation is reduced to a matrix equation with an infinite matrix:

$$L_{kmq,k'm'q'}\rho_{k'm'q'} = 0, \quad k,m,q = 0,\pm1,\pm2,...,\pm\infty, \tag{12}$$

where repeated indices imply summation. Matrix $L_{kmq,k'm'q'}$ can be represented as: $L = T + A + F - G$, where:

$$T_{kmq,k'm'q'} = ik\delta_{kk'}\delta_{mm'}\delta_{qq'},$$

$$A_{kmq,k'm'q'} = \delta_{kk'}\delta_{mm'}\frac{p_{\max}}{\pi}m \begin{cases} \frac{(-1)^{q-q'}}{q-q'}, & q \neq q', \\ 0, & q = q', \end{cases}$$

$$F = \mathbf{F}^{[t]}\mathbf{F}^{[\theta]}\mathbf{F}^{[p]}, \qquad F_{kk'}^{(t)} = \delta_{k',k+1} + \delta_{k',k-1},$$

$$F_{mm'}^{(\theta)} = \delta_{m',m+1} - \delta_{m',m-1}, \quad F_{qq'}^{(p)} = \delta_{qq'}\frac{f\pi}{4p_{\max}}q,$$

$$G_{kmq,k'm'q'} = \delta_{kk'}\delta_{mm'}\gamma \begin{cases} \frac{(-1)^{q-q'}}{q-q'}q', & q \neq q', \\ 0, & q = q'. \end{cases}$$

The boundary condition takes the form:

$$\sum_{q=-\infty}^{\infty} (-1)^q \rho_{kmq} = 0, \quad \forall\, k,m. \tag{13}$$

The normalization condition takes the form:

$$\rho_{k00} = 0, \quad \forall\, k \neq 0, \quad \rho_{000} = \frac{1}{4\pi p_{\max}}. \tag{14}$$

In the process of numerical calculation, the indexes are naturally limited to some integers, so that the matrices become finite-dimensional. Such limitation makes the Poincare cross section picture smoother, but in principle, the larger the boundary values of the indices, the more detailed the structure of the corresponding fractal can be obtained. We also note some features concerning the numerical implementation. First, the rapid increase in the size of the matrix with the growth of the boundary values of the indices. If the module of each index is limited to the

value of N, then the size of the matrix increases in order of magnitude as $(2N+1)^3 \times (2N+1)^3$. Secondly, the number of equations taking into account the boundary conditions and normalization conditions (equations (12–14) is greater than the number of unknown variables, so to find solutions it is necessary to use a method based on the minimization of some functional, for example, the least squares method. Third, the numerical solution of the system of equations (2) shows that the maximum value of the variable p, determined by the ratio (4) in real calculations is practically not achieved. For example, for the values of the parameters $f = 3$ and $\gamma = 0.1$ one obtains $p_{\max} = 30$, while for the real calculations of the system of equations (2) for sufficiently large time intervals p is limited to the value 4. Based on this, in numerical calculations using formulas (12–14), it is advisable to use a lower value for the value of $p_{\max}$.

Figur 3 shows the Poincare cross section pattern corresponding to $\tau = 0$, calculated with the parameters $f = 3$ and $\gamma = 0.1$ (left figure) and the probability density distribution pattern calculated using the method proposed above with the same values f, γ and boundary values of the indices $N = 17$. In the figures, the horizontal axis corresponds to the angle θ, and the vertical value p before scaling. As can be seen from the figures, the main features of the Poincare cross section are preserved in the calculation of the probability density.
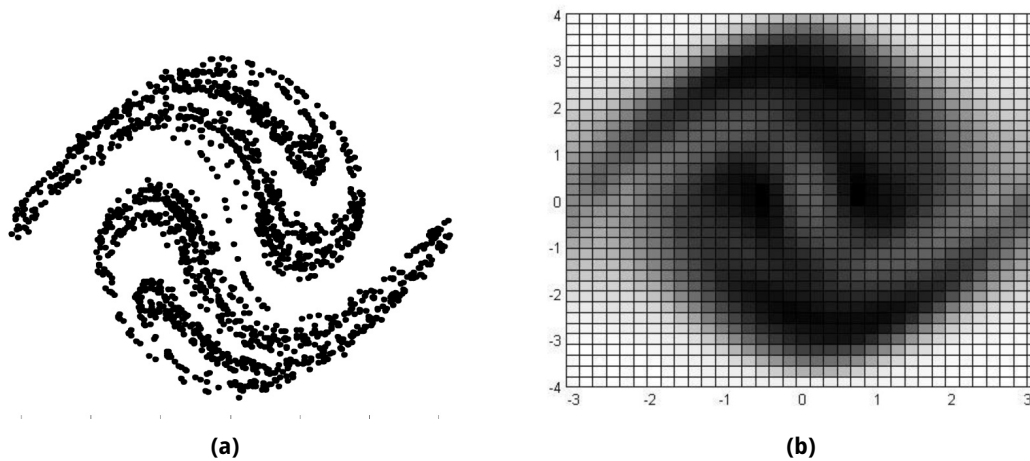

(a)   (b)

Figure 3

Note that the proposed method of calculation works only for chaotic States, that is, in cases when the trajectory of classical motion tends to a chaotic attractor at large time. For example, for the values of parameters $f = 2$ and $\gamma = 0.1$, the system motion becomes regular, so that the trajectory in the phase space tends to a closed curve, and the corresponding Poincare cross section contains only two points. At the same time, the calculation by the above method gives the picture shown in Fig. 4, resembling a picture at $f = 3$.

This feature is explained by the fact that the probability density distribution pattern is initially assumed to be periodic on the variables τ and θ. The Poincare cross section calculated in this case for arbitrary initial data does not have such periodicity.

2.3. Quantum problem and semiclassical limit

In accordance with the theory based on the classical consideration of the motion of the system under investigation here, the trajectory of the system in phase space tends to a chaotic attractor, and the Poincare cross section picture is a fractal. However, referring to the real physical
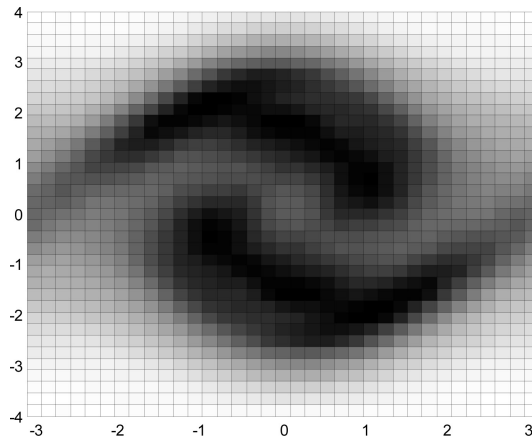
Figure 4

system, it is easy to understand that, since the phase space of the real system cannot be divided into arbitrarily small elements, the description using such an object as a fractal is approximate and adequate only as long as the distance between the points in the Poincare cross section is greater than the Planck constant $\hbar$. It is interesting to consider the corresponding quantum mechanical problem for this model and to make sure that the limit quasi-classical transition leads to the equations that were considered above.

As already mentioned, systems with dissipation are not Hamiltonian systems. To calculate the properties of such systems is not enough to consider the Schrödinger equation, it is necessary to use a more General method — the density matrix method (see, for example, [9]). The state of the system is determined by the density matrix, the equation for which has the form:

$$\dot{\rho} = -\frac{i}{\hbar}\left[H, \rho\right] + \Gamma\rho. \tag{15}$$

In this equation ρ — density matrix H — quantum Hamiltonian, Γ — relaxation matrix, the square brackets denote the commutator. The similarity of the notations for the classical and quantum cases should not cause confusion, in the quasi-classical limit between them there is an explicit correspondence.

The Hamiltonian of the system can be obtained from the classical Hamilton operator by following the usual quantization rules, that is, by replacing: p by the operator $\dfrac{\hbar}{i}\dfrac{\partial}{\partial\theta}$. Note that the eigenfunctions and eigenvalues of the Hamiltonian in the absence of external action ($f = 0$) are easily found:

$$\psi_m = \frac{1}{\sqrt{2\pi}}\exp(im\theta), \quad E_m = \frac{\hbar^2 m^2}{2}, \quad m = 0, \pm 1, \pm 2, \ldots$$

Further, we will use these eigenfunctions to decompose the density matrix. In the absence of relaxation ($\Gamma = 0$) the right part of the equation (15) takes the form:

$$-\frac{i}{\hbar}\left[H, \rho\right]_{m,m'} = -\frac{i\hbar}{2}(m^2 - m'^2)\rho_{m,m'} - \frac{if\cos t}{2\hbar}(\rho_{m+1,m'} + \rho_{m-1,m'} - \rho_{m,m'+1} - \rho_{m,m'-1}).$$

The term in equation (15) describing relaxation can be represented as ([9]):

$$\left(\Gamma\rho\right)_{mm'} = -(1-\delta_{mm'})\gamma_{mm'}\rho_{m,m'} + \delta_{mm'}\sum_{m''\neq m} W_{m,m''}\rho_{m''m''} \qquad (16)$$

Here the first term of the right part describes the relaxation of nondiagonal matrix elements of the density matrix, and the second one describes the relaxation of diagonal matrix elements. The parameters $W_{m,n}$ describe the transitions probability from level n to level m.

The relaxation parameters $\gamma_{m,n}, W_{m,n}$ in actual calculations, usually obtained from some model considerations. To carry out a quasi-classical transition, we make several assumptions about these parameters. Suppose, firstly, that only the probabilities of transition between neighboring levels are different from zero, and only the transition from a higher level to a lower one is possible, which corresponds to the low temperature limit. To find a match between the classical and quantum parameters, consider the change of the energy of the system in the absence of external influence ($f = 0$). In the classical case, taking into account the equations of motion, we obtain:

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{p^2}{2}\right) = -\gamma p^2.$$

In the quantum case:

$$\frac{dE}{dt} = W_{m,m+1}(E_m - E_{m+1}) = -W_{m,m+1}\hbar^2 m \qquad (17)$$

Since the classical momentum p corresponds to the value $\hbar m$, to match the quantum relaxation to the classical force of viscous friction is enough to put:

$$W_{m,m+1} = \gamma m. \qquad (18)$$

As a result, from the expression (16) we obtain:

$$\left(\Gamma\rho\right)_{mm'} = -(1-\delta_{mm'})\gamma_{mm'}\rho_{m,m'} + \delta_{mm'}\gamma\big((m+1)\rho_{m+1,m+1} - m\rho_{m,m}\big).$$

As a second assumption, we assume that the relaxation of nondiagonal matrix elements is described similarly to the relaxation of diagonal elements, that is, we assume that for diagonal and nondiagonal matrix elements, the action of the relaxation matrix can be written as:

$$\left(\Gamma\rho\right)_{mm'} = \gamma\left(\left(\frac{m+m'}{2}+1\right)\rho_{m+1,m'+1} - \frac{m+m'}{2}\rho_{m,m'}\right).$$

As a result of these assumptions, the equation (15) for the density matrix takes the form:

$$\dot{\rho}_{m,m'} = -\frac{i\hbar}{2}\big(m^2 - m'^2\big) - \frac{if\cos t}{2\hbar}\big(\rho_{m+1,m'} + \rho_{m-1,m'} - \rho_{m,m'+1} - \rho_{m,m'-1}\big)+$$

$$+\gamma\left(\left(\frac{m+m'}{2}+1\right)\rho_{m+1,m'+1} - \frac{m+m'}{2}\rho_{m,m'}\right).$$

The quasi-classical limit corresponds to large values of quantum numbers m. For the transition to the quasi-classical limit, it is advisable to use the Wigner representation for the density matrix, which allows a convenient transition to the classical limit (see, for example, [10]). Define the variables:

$$P = \frac{m+m'}{2}, \quad Q = m - m',$$

and we will denote the density matrix in this representation as $\rho(P,Q)$. The equation for the density matrix takes the form:

$$\dot{\rho}(P,Q) = -i\hbar PQ-$$

$$-\frac{if\cos t}{2\hbar}\Big(\rho\big(P+1/2,Q+1\big) + \rho\big(P-1/2,Q-1\big) - \rho\big(P+1/2,Q-1\big) - \rho\big(P-1/2,Q+1\big)\Big)+$$

$$+\gamma\big((P+1)\rho(P+1,Q) - P\rho(P,Q)\big).$$

The quasi-classical limit corresponds to large values of the variable P, but the variable Q can be small (of the order of one). Given the large values of P, we can proceed to the derivatives based on the formula:

$$\rho(P+1,Q) \approx \rho(P,Q) + \frac{\partial\rho(P,Q)}{\partial P}.$$

Considering also that

$$\rho(P,Q) << P\frac{\partial\rho(P,Q)}{\partial P},$$

we obtain the equation for the density matrix:

$$\dot{\rho}(P,Q) = -i\hbar PQ\rho(P,Q) - \frac{if\cos t}{2\hbar}\left(\frac{\partial\rho(P,Q+1)}{\partial P} - \frac{\partial\rho(P,Q-1)}{\partial P}\right) + \gamma P\frac{\partial\rho(P,Q)}{\partial P}. \qquad (19)$$

The transition to the classical equation is carried out using the Fourier transform (see [10]):

$$\rho(P,\theta) = \frac{1}{2\pi}\int \rho(P,Q)\exp(iQ\theta)d\theta.$$

Applying the inverse Fourier transform, it is easy to obtain the equation for $\rho(P,\theta)$:

$$\dot{\rho}(P,\theta) = -\hbar P\frac{\partial\rho(P,\theta)}{\partial\theta} - \frac{f\cos t\sin\theta}{\hbar}\frac{\partial\rho(P,\theta)}{\partial P} + \gamma P\frac{\partial\rho(P,\theta)}{\partial P}. \qquad (20)$$

For the final transition from equation (20) to equation (8), it is sufficient to associate the variable P corresponding to quantum numbers with the macroscopic variable p:

$$p = \hbar P.$$

The equivalence of the classical equation (8) for the distribution of the probability density of states and the quantum mechanical equation (15) for the density matrix suggests that, if based on the probabilistic approach, the "predictability" of the classical equation (8) is the same as in the case of the quantum mechanical problem. We can calculate with the same degree of confidence the probability that the state of the system lies in a certain region of the phase space during chaotic motion. The characteristic features of classical chaos are the loss of determinism, that is, the impossibility of predicting the state of the system through the time interval of the order λ^{-1}, where λ is the maximal Lyapunov exponent, as well as fractals formed in Poincare cross section. However, in the quantum problem, both the determinism of the trajectory of motion and fractals are principal absent. There is a natural question — whether the concept of chaos, at least for the considered problem, is connected with the fact that we are trying to apply the classical description, which is essentially approximate, to describe the movement of the system? Note that unlike dynamic equations (2), the equation for the probability density distribution (8) is linear. We also note that the concept of "quantum chaos" is usually associated

(see, for example, [4, 11]) with features in the spectra of systems in the transition to the quasi-classical limit. The question of whether there are such features in the spectra of the problem under consideration requires further investigation.

In conclusion, we note that in this problem the equation (8) for the probability density distribution was derived from the Hamilton equations for the mechanical system. However, the connection of the first-order partial differential equation with the system of differential equations for characteristic curves allows, at least formally, to derive an equation for the probability density distribution of any chaotic system described by a system of ordinary differential equations, for example, for the Lorentz system. The question of whether such an equation will really describe the probability density distribution for the chaotic state of an arbitrary system also requires further investigation.

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